Calculus III — Pretest #3 — Name: ______________________

Fall, 2005 — Dr. D. P. Story

Global Instructions: Test #13 will be Friday, November 18, 2005. The test covers the following sections in the text: §§15.6–15.7; §§16.1–16.6. The following are a selection of problems from the material to be covered on the test. These problems *do not* represent the entirely of the types of problems that may appear on the test. Solve these problems, ideally, without reference to your text.

(18^{pts}) **1.** Do each of the following.

(a) *Solve* the iterated integral: $\displaystyle\int_0^1\int_{x^3}^{2x} xy\;dy\,dx$. (*Note:* Continue on on the back of page 1, as needed.)

Solution:

$$\begin{aligned}\int_0^1\int_{x^3}^{2x} xy\,dy\,dx &= \frac{1}{2}\int_0^1 xy^2\Big|_{y=x^3}^{y=2x}\,dx = \frac{1}{2}\int_0^1 x(4x^2-x^6)\,dx\\ &= \frac{1}{2}\int_0^1 4x^3-x^7\,dx = \frac{1}{2}\left[\left(x^4-\frac{1}{8}x^8\right)\Big|_0^1\right]\\ &= \frac{1}{2}(1-\frac{1}{8}) = \frac{1}{2}\frac{7}{8} = \boxed{\frac{7}{16}}\end{aligned}$$

(b) Sketch the region of integration on on the back of page 1. (Be sure to shade in the region, and label the boundary curves.)

Solution: Left to the student.

(8^{pts}) **2.** Sketch the solid whose volume is given by the iterated integral: $\displaystyle V=\int_0^1\int_0^{1-x} 1-x-y\,dy\,dx.$

Solution: Left to the student.

($8^{\text{pts}}_{\text{ea.}}$) **3.** Consider the problem of calculating the integral $\displaystyle\iint_D f(x,y)\,dA$, where D is the region bounded by $y=x+6$ and $y=x^2$.

(a) Taking $dA=dy\,dx$, *set up* the limits of integration;

Solution: The two curves intersect at $x=-2$ and $x=3$. The limits of integration then are

$$\iint_D f(x,y)\,dA = \int_{-2}^3\int_{x^2}^{x+6} f(x,y)\,dy\,dx$$

(b) Taking $dA=dx\,dy$, *set up* the limits of integration.

Solution: The two curves intersect at $y=4$ and $y=9$. We must break the region up into two regions.

$$\iint_D f(x,y)\,dA = \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy + \int_4^9\int_{y-6}^{\sqrt{y}} f(x,y)\,dx\,dy$$

(8^{pts}) **4.** Reverse the order of integration: $\displaystyle\int_0^1\int_{3y^2}^3 e^{x^2}\,dx\,dy.$

Solution: Here, you must sketch the region of integration to correctly reverse the order.

$$\int_0^1\int_{3y^2}^3 e^{x^2}\,dx\,dy = \int_0^3\int_0^{\sqrt{x/3}} e^{x^2}\,dy\,dx$$

(8^{pts}) **5.** Consider the integral: $\iint_D \frac{1}{(1+x^2+y^2)^{3/2}}\,dA$. *Set up* the limits of integration for this integral by changing to polar coordinate variables. Here, D is the region in the first quadrant bounded by $y = x$, $x^2 + y^2 = 4$ and the y–axis.

Solution:

$$\iint_D \frac{1}{(1+x^2+y^2)^{3/2}}\,dA = \int_{\pi/4}^{\pi/2} \int_0^2 \frac{1}{(1+r^2)^{3/2}}\, r\,dr\,d\theta$$

(8^{pts}) **6.** *Set up* the double integral for the *volume* of the solid below the graph of the function $f(x, y) = x^2 + y^2$, and above the region D in the first quadrant of the xy–plane bounded by $x = 0$, $y = 0$, and $y = 4 - 2x$.

Solution: After drawing a picture of the situation we see that

$$V = \int_0^2 \int_0^{4-2x} x^2 + y^2\,dy\,dx$$

(10^{pts}) **7.** *Set up* the double integral for the moment about the y–axis for the region D in the first quadrant bounded by the parabola $y = x^2$ and the line $y = x$ given that the density function is $\rho(x, y) = x + y$.

Solution:

$$M_y = \iint_D x\rho(x,y)\,dA = \int_0^1 \int_{x^2}^{x} x(x+y)\,dy\,dx$$

(10^{pts}) **8.** Consider the following problem:

Maximize/Minimize: $f(x, y, z) = x + 2y$

Subject to: $x + y + z = 1$

$x^2 + z^2 = 1$

Set up the Lagrange system of equations corresponding to this max/min problem.

Solution: This is a two constraint problem. Let $g(x, y, z) = x + y + z$ and $h(x, y, z) = x^2 + z^2$. The system is

$$\begin{aligned} \nabla f &= \lambda \nabla g + \mu \nabla h \\ x + y + z &= 1 \\ x^2 + z^2 &= 1 \end{aligned}$$

If we calculate the gradients of f, g and h, and make the first (vector) equation into three scalar equations, we obtain

$$\begin{aligned} 1 &= \lambda + 2\mu x \\ 2 &= \lambda \\ 0 &= \lambda + 2\mu z \\ x + y + z &= 1 \\ x^2 + z^2 &= 1 \end{aligned}$$

This is the desired answer.

(14^{pts}) **9.** Consider the problem of calculating the *surface area* of that portion of the graph of $f(x, y) = 9 - x^2 - y^2$ for (x, y) belonging to the region $D = \{\,(x, y) \mid x^2 + y^2 \le 9\,\}$.

(a) *Set up* the integral necessary to solve this problem.

Solution: $f_x = -2x$, $f_y = -2y$. The set up is

$$\begin{aligned} S &= \iint_D \sqrt{4x^2 + 4y^2 + 1}\,dA = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{4x^2 + 4y^2 + 1}\,dy\,dx \\ &= \int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1}\, r\,dr\,d\theta \end{aligned}$$

(b) Now, *calculate* the integral.

Solution:

$$S = \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\, r\,dr\,d\theta = 2\pi \int_0^3 (4r^2+1)^{1/2}\, r\,dr$$
$$= 2\pi \frac{1}{8}\int_0^3 (4r^2+1)^{1/2}\, 8r\,dr = \frac{\pi}{4}\frac{2}{3}\,(4r^2+1)^{3/2}\Big|_0^1$$
$$= \frac{\pi}{6}[\,5^{3/2}-1\,] = \boxed{\frac{\pi}{6}[\,5\sqrt{5}-1\,]}$$

(10^{pts}) **10.** Consider the function $f(x,y) = x^3 - 3xy + 8y^3$. Find and classify all critical points.

Solution: Find all first partials: $f_x = 3x^2 - 3y$ and $f_y = -3x + 24y^2$. Set up equations:

$$3x^2 - 3y = 0 \tag{1}$$
$$-3x + 24y^2 = 0 \tag{2}$$

Now, solving for y in equation (1) and substituting into (2), and after cancelling out common factors, we get $-x + 8x^4 = 0$. This equation has two solutions: $x = 0$ and $x = \frac{1}{2}$; hence, there are two critical points, $(x,y) = (0,0)$ and $(\frac{1}{2}, \frac{1}{4})$.

Now to classify, we need to compute the *discriminant*: $D(x,y) = f_{xx}f_{yy} - f_{xy}^2$. Calculations: $f_{xx} = 6x$, $f_{yy} = 48y$ and $f_{xy} = -3$. Thus,

$$D(x,y) = (6x)(48y) - 9 = 6(48)xy - 9 = 288xy - 9$$

Finally

(x,y)	$D(x,y)$	$f_{xx}(x,y)$	Classification
$(0,0)$	$(-)$	n/a	Saddle point
$(\frac{1}{2},\frac{1}{4})$	$(+)$	$(+)$	minimum