Calculus III	Pretest #2	Name:________________
Fall, 2005		Dr. D. P. Story

Global Instructions: Test #2 will be Friday, October 28, 2005. The test covers the following sections in the text: §§14.3,14.4; §§15.1–15.6. The following are a selection of problems from the material to be covered on the test. These problems *do not* represent the entirely of the types of problems that may appear on the test. Solve these problems, ideally, without reference to your text.

1. The velocity of a particle at time t is given by $\vec{v}(t) = 2t^2\vec{\imath} - 3t\vec{\jmath}$.

(a) What is the *speed* of the particle at time $t = 2$?

$\boxed{\text{speed} = 10}$

Solution: speed $= |\vec{v}(t)| = \sqrt{4t^4 + 9t^2} = |t|\sqrt{4t^2 + 9}$. Now, at time $t = 2$, the speed is $\boxed{\text{speed} = 10}$.

(b) What is the *acceleration* of the particle at time $t = 2$?

$\boxed{\text{accel} = 8\vec{\imath} - 3\vec{\jmath}}$

Solution: $\vec{a}(t) = \vec{v}\,'(t) = \boxed{4t\vec{\imath} - 3\vec{\jmath}}$; at time $t = 2$, $\boxed{\text{accel} = \vec{a}(2) = 8\vec{\imath} - 3\vec{\jmath}}$.

2. Find the velocity and position vectors of a particle given the following information: $\vec{a}(t) = \vec{k}$, $\vec{v}(0) = \vec{\imath} - \vec{\jmath}$, and $\vec{r}(0) = \vec{0}$.

Solution:

$$\vec{v}(t) = \int \vec{a}(t)\,dt = t\vec{k} + \vec{C}$$

But, $\vec{\imath} - \vec{\jmath} = \vec{v}(0) = \vec{C}$, says that $\vec{C} = \vec{\imath} - \vec{\jmath}$. So,

$$\boxed{\vec{v}(t) = \vec{\imath} - \vec{\jmath} + t\vec{k}}$$

Similarly,

$$\vec{r}(t) = \int \vec{v}(t)\,dt = t\vec{\imath} - t\vec{\jmath} + \tfrac{1}{2}t^2\vec{k} + \vec{C}$$

Again, $\vec{0} = \vec{r}(0) = \vec{C}$, so $\vec{C} = \vec{0}$. Thus

$$\boxed{\vec{r}(t) = t\vec{\imath} - t\vec{\jmath} + \tfrac{1}{2}t^2\vec{k}}$$

3. On back of page 1, make a sketch of the domain of the function: $f(x, y) = \dfrac{\ln(x^3 - y)}{\sqrt{x}}$.

Solution: Left to the reader, that's you.

4. Argue that $\lim_{(x,y)\to(0,0)} \dfrac{3xy}{4x^2+5y^2}$ does not exist.

Solution: Approach $(0,0)$ along two distinct paths. $x=0$ and $y=x$. Details left to the reader, that's you.

5. Perform the indicated calculations.

(a) $f(x,y)=\sqrt{x^2y+4}$, calculate $f_x(-2,3)=$

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Solution:

$$f_x(x,y)=\frac{2xy}{2\sqrt{x^2y+4}}=\frac{xy}{\sqrt{x^2y+4}}$$

and so $\boxed{f_x(-2,3)=-6/4=-3/2}$.

(b) $z=e^{xy}\sin(xy)$, calculate $\dfrac{\partial z}{\partial y}=$

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Solution:

$$\begin{aligned}\frac{\partial z}{\partial y}&=e^{xy}\frac{\partial}{\partial y}\sin(xy)+\frac{\partial e^{xy}}{\partial y}\sin(xy)=xe^{xy}\cos(xy)+xe^{xy}\sin(xy)\\&=\boxed{xe^{xy}(\sin(xy)+\cos(xy))}\end{aligned}$$

6. Let $w=f(x,y,z)$ and $\begin{cases}x=g(s,t)\\y=h(s,t)\\z=q(s,t)\end{cases}$. Use the *chain rule* to write a formula for $\partial z/\partial t$.

$\dfrac{\partial w}{\partial t}=$

Solution:

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

7. Consider the function $f(x,y)=3x^4y^2$. Find the *equation* of the *plane tangent* to the graph of $f(x,y)$ at the point on the surface corresponding to $(x,y)=(1,-1)$.

Solution: Preliminary calculations: $f(1,-1)=3$, $f_x(x,y)=12x^3y^2$, $f_y(x,y)=6x^4y$, $f_x(1,-1)=12$, $f_y(1,-1)=-6$

$$12(x-1)-6(y+1)-(z-3)=0$$

$$\boxed{12x-6y-z=15}$$

8. Calculate $\dfrac{\partial z}{\partial x}$, where $3yz^2-x=y\sin(xz)$. (Continue on the back of page 1 as needed.)

$\dfrac{\partial z}{\partial x}=$

Solution: Let $F(x,y,z)=3yz^2-x-y\sin(xz)$, then

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{-1-yz\cos(xz)}{6yz-xy\cos(xz)}=\boxed{\frac{1+yz\cos(xz)}{6yz-xy\cos(xz)}}$$

9. Let $f(x,y) = \dfrac{y}{x+y}$. It is easy to calculate $\nabla f(x,y) = \dfrac{\langle -y, x\rangle}{(x+y)^2}$. Using this fact, calculate $D_{\vec{u}} f(1,2)$, where $\vec{u}$ is the direction defined from $P(1,2)$ to $Q(4,6)$.
Solution: $\overrightarrow{PQ} = \langle 3,4\rangle$, now we make this vector into a unit vector $\vec{u} = \frac{1}{5}\langle 3,4\rangle$. Also, $\nabla f(1,2) = \frac{1}{9}\langle -2,1\rangle$, and so

$$D_{\vec{u}} f(1,2) = \nabla f(1,2)\cdot \vec{u} = \tfrac{1}{9}\langle -2,1\rangle \cdot \tfrac{1}{5}\langle 3,4\rangle = \tfrac{1}{45}(-6+4) = \boxed{-\tfrac{2}{45}}$$

10. Consider the function $f(x,y) = x^3(2y-1)^2$.

(a) (4 pts) Calculate the maximum directional derivative of f at the point $(1,1)$.
Solution: Calculate the gradient first $\nabla f(x,y) = \langle 3x^2(2y-1)^2, 4x^3(2y-1)\rangle$; and at $(1,1)$ we have $\nabla f(1,1) = \langle 3,4\rangle$. The maximal directional derivative is then

$$\max D_{\vec{u}} f(1,1) = |\nabla f(1,1)| = |\langle 3,4\rangle| = \boxed{5}$$

(b) (3 pts) In what direction is the directional derivative at the point $(1,1)$ at its maximum? (Leave you answer as a unit vector $\vec{u}$.)
Solution:

$$\boxed{\vec{u} = \tfrac{1}{5}\langle 3,4\rangle}$$

11. The volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$. The radius and height are measured to be $r = 1.3$ and $h = 4.1$, but the values are only accurate to $\pm .1$. Use differentials to estimate the error in the calculation of V.
Solution: $dV = \frac{1}{3}\pi(2rh\,dr + r^2\,dh) = \frac{1}{3}\pi r(2h\,dr + r\,dh)$, thus,

$$\begin{aligned} |dV| &\le \tfrac{1}{3}\pi(1.3)(2(4.1)(.1) + (1.3)(.1)) \\ &\le \tfrac{1}{3}\pi(1.3)(2(4.1) + 1.3)(.1) \\ &\le \tfrac{1}{3}\pi(1.3)(9.5)(.1) \\ &\le \boxed{1.293287883333} \end{aligned}$$

12. Consider the equation $x^2 z + z^2 y^3 = z + 1$.

(a) Find the vector that is normal to the surface at $(1,1,1)$.
Solution: Let $F(x,y,z) = x^2 z + z^2 y^3 - z - 1$, then

$$\nabla F(x,y,z) = \langle 2xz, 3z^2y^2, x^2 + 2zy^3 - 1\rangle$$

A normal vector is $\vec{n} = \nabla F(1,1,1) = \boxed{\langle 2,3,2\rangle}$

(b) Find the equation of the plane tangent to the surface $x^2 z + z^2 y^3 = z + 1$ at at $(1,1,1)$.

Solution: From the previous part, a normal vector is $\vec{n} = \nabla F(1,1,1) = \boxed{\langle 2,3,2\rangle}$. The plane passes through $(1,1,1)$, so, an equation of the tangent plane is...

$$\boxed{2(x-1) + 3(y-1) + 2(x-1) = 0}$$

and this can be put into general form.

13. Let $f(x, y, z) = x^3y^4z^5 + xyz$. Verify that $f_{xxz} = f_{xzx}$.

(a) Calculation of f_{xxz}

Solution: Calculations:

$$f_x = 3x^2y^4z^5 + yz$$
$$f_{xx} = 6xy^4z^5$$
$$f_{xxz} = 30xy^4z^4$$

(b) Calculation of f_{xzx}

Solution: Calculations:

$$f_x = 3x^2y^4z^5 + yz$$
$$f_{xz} = 15x^2y^4z^4 + y$$
$$f_{xzx} = 30xy^4z^4$$

Verified!