

Instructions. Test #1 will be Monday, September 26, 2005. The test covers the following sections in the text: §§13.1-13.6; §§14.1,14.2. The following are a selection of problems from the material to be covered on the test. These problems *do not* represent the entirety of the types of problems that may appear on the test. Solve these problems, ideally, without reference to your text.

(3^{pts}_{ea.}) 1. Let $\vec{a} = \langle 2, -2, 1 \rangle$, $\vec{b} = \langle 1, -2, 1 \rangle$, $P(2, -3, 2)$, and $Q(-1, -2, 3)$. Calculate each of the following, show your work.

(a) $-3\vec{a} + 2\vec{b} =$

Solution:

$$-3\vec{a} + 2\vec{b} = -3\langle 2, -2, 1 \rangle + 2\langle 1, -2, 1 \rangle = \langle -6, 6, -3 \rangle + \langle 2, -4, 2 \rangle = \langle -4, 2, -1 \rangle$$

(b) $|\vec{b} - \vec{a}| =$

Solution: We have $\vec{b} - \vec{a} = \langle 2, -2, 1 \rangle - \langle 1, -2, 1 \rangle = \langle 1, 0, 0 \rangle$ and so $|\vec{b} - \vec{a}| = 1$

(c) the *direction cosines* of \vec{a} :

Solution: We have that $|\vec{a}| = 3$. And so,

$$u = \frac{1}{3}\langle 2, -2, 1 \rangle$$

Thus $\cos \alpha = 2/3$, $\cos \beta = -2/3$ and $\cos \gamma = 1/3$.

(d) $\vec{QP} = \langle \quad, \quad, \quad \rangle$

Solution: We compute "head minus tail":

$$\vec{QP} = \langle 2, -3, 2 \rangle - \langle -1, -2, 3 \rangle = \langle 3, -1, -1 \rangle$$

(e) Let $\theta =$ the angle between \vec{a} and \vec{b} . Then $\cos \theta =$

Solution: The standard calculations are as follows:

$$a \cdot b = 7 \quad |a| = 3 \quad |b| = \sqrt{6}$$

$$\cos \theta = \frac{7}{3\sqrt{6}}$$

(f) Put the tail of \vec{b} at the point Q , and calculate the coordinates of the head of the vector.

head =

Solution: Simply add the components of \vec{b} to the coordinates of Q .

$$\text{head} = (-1 + 1, -2 - 2, 3 + 1) = \langle 0, -4, 4 \rangle$$

Alternately, we could have simply calculated the vector $\vec{Q} + \vec{b}$.

- (5pts) **2.** Calculate the *area* of the *triangle* formed by putting the vectors $2\vec{j}$ and $2\vec{i} - 3\vec{k}$ tail-to-tail.

area=

Solution:

$$2\vec{j} \times (2\vec{i} - 3\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 0 \\ 2 & 0 & -3 \end{vmatrix} = -6\vec{i} - 4\vec{k} = -2(3\vec{i} + 2\vec{k})$$

and so

$$\text{area} = \frac{1}{2} |2\vec{j} \times (2\vec{i} - 3\vec{k})| = \frac{1}{2} 2 |3\vec{i} + 2\vec{k}| = \boxed{\sqrt{13}}$$

- (8pts) **3.** Given the vectors $\vec{v} = \langle -1, 1, 3 \rangle$ and $\vec{u} = \langle 2, -1, 2 \rangle$, find a vector that is perpendicular to both \vec{v} and \vec{u} .

Solution: We calculate the cross product:

$$\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 3 \\ 2 & -1 & 2 \end{vmatrix} = 5\vec{i} + 8\vec{j} - \vec{k} = \boxed{\langle 5, 8, -1 \rangle}$$

- (8pts) **4.** Consider $\vec{a} = \langle -2, 1, 3 \rangle$ and $\vec{b} = \langle -1, 2, 1 \rangle$. Calculate two vectors $\vec{\omega}_1$ and $\vec{\omega}_2$ such that $\vec{b} = \vec{\omega}_1 + \vec{\omega}_2$ where $\vec{\omega}_1 \parallel \vec{a}$ and $\vec{\omega}_2 \perp \vec{a}$.

$$\vec{\omega}_1 = \langle \quad, \quad, \quad \rangle \quad \vec{\omega}_2 = \langle \quad, \quad, \quad \rangle$$

Solution: First calculate $\vec{\omega}_1 = \text{Proj}_{\vec{a}} \vec{b}$. Indeed,

$$\vec{a} \cdot \vec{b} = (-2)(-1) + (1)(2) + (3)(1) = 2 + 2 + 3 = 7$$

$$\vec{a} \cdot \vec{a} = (-2)^2 + (1)^2 + (3)^2 = 4 + 1 + 9 = 14$$

$$\text{Proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \cdot \vec{a} = \frac{7}{14} \cdot \langle -2, 1, 3 \rangle = \frac{1}{2} \cdot \langle -2, 1, 3 \rangle = \langle -1, \frac{1}{2}, \frac{3}{2} \rangle$$

Thus

$$\vec{\omega}_1 = \langle -1, \frac{1}{2}, \frac{3}{2} \rangle$$

and finally

$$\vec{\omega}_2 = \vec{b} - \vec{\omega}_1 = \langle -1, 2, 1 \rangle - \langle -1, \frac{1}{2}, \frac{3}{2} \rangle = \langle 0, \frac{3}{2}, -\frac{1}{2} \rangle$$

$$\vec{\omega}_2 = \langle 0, \frac{3}{2}, -\frac{1}{2} \rangle$$

- (8pts) **5.** Write the parametric equations of the *line* passing through the point $P(-1, -2, 3)$ and perpendicular to the plane $-2x - 3y + 2z + 5 = 0$.

$$\begin{cases} x = \\ y = \\ z = \end{cases}$$

Solution: A vector normal to the given plane is $\vec{n} = \langle -2, -3, 2 \rangle$. This vector is parallel to the line we wish to construct.

$$\begin{cases} x = -1 - 2t \\ y = -2 - 3t \\ z = 3 + 2t \end{cases}$$

- (8pts) 6. Consider the plane $3x - 2y + 5z = 0$ and the line $x = -1 + 2t$, $y = 2 - t$, $z = 4t$. Find the point, P , where the line *intersects* the plane.

$$P(\quad , \quad , \quad)$$

Solution: We have $3(-1 + 2t) - 2(2 - t) + 5(4t) = 0$, or $28t = 7$, or $t = \frac{1}{4}$. Substituting this value for t into the parametric equations, we get the point of intersection is $P(-\frac{1}{2}, \frac{7}{4}, 1)$

- (8pts) 7. Consider the two planes $x + y + x = 1$ and $x - 3y + z = 2$. Calculate the cosine of the acute angle these two planes make with each other.

$$\cos \theta =$$

Solution: Vectors which are normal to the two planes are $\vec{n}_1 = \langle 1, 1, 1 \rangle$ and $\vec{n}_2 = \langle 1, -3, 1 \rangle$. Now $\vec{n}_1 \cdot \vec{n}_2 = -1$, the angle between these two vectors is acute.

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{-1}{\sqrt{3} \sqrt{11}}$$

The answer is

$$\cos \theta = \frac{-1}{\sqrt{3} \sqrt{11}}$$

- (8pts) 8. Find the parametric equations that describe the *line tangent* to the graph of $\vec{r}(t) = (2t^2 + 1)\vec{i} - (1 + e^{3t})\vec{j} + \sin 3t\vec{k}$ at the point on the graph corresponding to $t = 0$.

$$\begin{cases} x = \\ y = \\ z = \end{cases}$$

Solution: Note that $\vec{r}(0) = \vec{i} - 2\vec{j} = \langle 1, -2, 0 \rangle$. Also, $\vec{r}'(t) = 4t\vec{i} - 3e^{3t}\vec{j} + 3\cos 3t\vec{k}$ and $\vec{r}'(0) = -3\vec{j} + 3\vec{k} = \langle 0, -3, 3 \rangle$. Thus,

$$\begin{cases} x = 1 \\ y = -2 - 3t \\ z = 3t \end{cases}$$

is an equation of the desired line.

(8pts) 9. Let the position of a particle at time t is given by $\vec{r}(t) = \langle t^3, 5t^2, 6t \rangle$.

(a) Calculate $\vec{r}'(t)$

$$\vec{r}'(t) = \langle \quad, \quad, \quad \rangle$$

Solution: Just differentiate!

$$\vec{r}'(t) = \langle 3t^2, 10t, 6 \rangle$$

(b) Calculate $\vec{r}''(t)$

$$\vec{r}''(t) = \langle \quad, \quad, \quad \rangle$$

Solution: Just differentiate!

$$\vec{r}''(t) = \langle 6t, 10, 0 \rangle$$

(c) Calculate $\vec{T}(t)$, the *unit tangent vector*, at $t = 1$.

$$\vec{T}(1) = \langle \quad, \quad, \quad \rangle$$

Solution: Calculate the unit vector of $\vec{r}'(1)$.

$$\vec{r}'(1) = \langle 3, 10, 6 \rangle \quad |\vec{r}'(1)| = \sqrt{145}$$

$$\vec{T}(1) = \frac{1}{\sqrt{145}} \langle 3, 10, 6 \rangle$$

(8pts) 10. Let $\vec{r}(t)$ satisfy the conditions $\vec{r}'(t) = 2e^{3t}\vec{i} + 3e^{-t}\vec{j}$ and $\vec{r}(0) = 3\vec{i}$. Find $\vec{r}(t)$.

$$\vec{r}(t) = \langle \quad, \quad, \quad \rangle$$

Solution:

$$\vec{r}(t) = \int \vec{r}'(t) dt = \int 2e^{3t} dt \vec{i} + \int 3e^{-t} dt \vec{j} = \frac{2}{3}e^{3t}\vec{i} - 3e^{-t}\vec{j} + \vec{C} \quad (1)$$

$$\vec{r}(0) = \frac{2}{3}\vec{i} - 3\vec{j} + \vec{C} \quad (2)$$

But we require $\vec{r}(0) = 3\vec{i}$, equating this with equation (2), we get $\frac{2}{3}\vec{i} - 3\vec{j} + \vec{C} = 3\vec{i}$, from which we deduce $\vec{C} = \frac{7}{3}\vec{i} + 3\vec{j}$. Finally,

$$\vec{r}(t) = \frac{2}{3}e^{3t}\vec{i} - 3e^{-t}\vec{j} + \frac{7}{3}\vec{i} + 3\vec{j} = \frac{1}{3}(2e^{3t} + 7)\vec{i} - 3(e^{-t} - 1)\vec{j}$$

(8^{pts}) **11.** Consider the hyperboloid $\frac{x^2}{4} - y^2 - \frac{z^2}{16} = 1$.

(a) Find the equation of the trace of this surface onto the xz -plane.

equation:

Solution: We put $y = 0$ in the equation to obtain

$$\frac{x^2}{4} - \frac{z^2}{16} = 1$$

(b) Sketch the surface of the hyperboloid on back of on the back of page 4.

Solution: Left to the student.

(5^{pts}) **12.** Sketch the graph of the vector-valued function $\vec{r}(t) = \langle \cos t, t, \sin t \rangle$. [Do the sketch on on the back of page 4.]

Solution: Left to the student.