LINEAR PREDICTION OF BANDPASS SIGNALS BASED ON PAST SAMPLES

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ABSTRACT

This paper describes methods of computation of prediction coefficients for bandpass signals which provide highly accurate predictions from past samples of a signal based on the sampling rate, the bandwidth, and the center frequency of the signal. Two methods are described, one based on solving a matrix system and another on eigenvectors related to and extending the Discrete Prolate Spheroidal Sequences. In contrast to the standard LPC method, the resulting prediction coefficients are independent of the prediction time. Several examples are given.

1. INTRODUCTION

Linear prediction of the next value of a signal $x(t)$ based on its past samples is an important problem with applications in signal compression and other areas. Linear prediction has the form

$$x(t) = \sum_{n=1}^{N} a_n x(t - nT), \qquad (1)$$

where the constants $\{a_n\}_{n=1,\ldots,N}$ are the prediction coefficients and T is the sampling interval. Previous work of the authors, [3] [4] [5] [6] [7], has established methods of prediction for band-limited signals which work well when the Nyquist rate is observed. In this paper, the authors present new results that extend the methods for linear prediction to bandpass signals. It will be shown that the solution of a Toeplitz system involving the bandwidth, sampling interval T, and center frequency of the passband leads to prediction coefficients that are optimal for this class of signals. Based on the known structure of the bandpass signal, it will be shown that these are the only parameters needed to obtain a set of prediction coefficients which applies to the entire class of such signals.

2. BANDPASS SIGNALS AND SAMPLING

The famous sampling theorem (Whittaker-Kotel'nikov-Shannon) has an extension to the bandpass case [8] In this paper, it will be shown that linear prediction for the bandpass case is likewise a similar extension of previous prediction methods for band-limited signals.

Suppose that the Fourier representation of the signal is $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$, where $X(f)$ is a finite energy signal. If $X(f) = 0$ when $|f \pm f_c| > B/2$, then we say that $x(t)$ is a bandpass signal of bandwidth B at center frequency f_c. Because of the applications, it is often assumed that the signal $x(t)$ is real-valued, in which case $X(f)$ is Hermitian, i.e. $X(-f) = \overline{X(f)}$. However, for the general development here, we assume the case of a complex-valued signal and suppose that the Fourier representation of $x(t)$ is in the union of the intervals $(-W_2, -W_1)$ and (W_1, W_2), where $W_1 := f_c - B/2$ and $W_2 := f_c + B/2$. This admits, for example, the function $x(t) = e^{j2\pi f_c t}$, into the following discussion.

The extended sampling theorem [8], p.281, gives the form of signal $x(t)$ as

$$k \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}(B(t - nT)) \cos(2\pi f_c(t - nT)), \quad (2)$$

with constant $k := 2TB/\pi$ and with the usual definition of $\text{sinc}(t)$ as $\sin(\pi t)/(\pi t)$. The Nyquist criterion for the bandpass signal is less restrictive in the size of the sampling interval T. Sampling for bandpass signals allows for larger T than a criterion that ignored the particular nature of the spectrum of the signal and based the sampling on criterion for band-limited signals.

3. OPTIMAL PREDICTION FOR BANDPASS SIGNALS

In this paper, optimal prediction coefficients will be derived for bandpass signals. We first define a function that modulates the standard sinc function to the bandpass interval, by defining

$$\text{csinc}(t) := \text{sinc}(Bt)\cos(2\pi f_c t), \tag{3}$$

where B is the bandwidth and f_c is the center frequency, as used in the bandpass sampling theorem (2).

The optimal prediction coefficients considered in this paper are those which minimize the error expression $|x(t)-\sum_{n=1}^{N} a_n x(t-nT)|$ over the class of bandpass signals as described above.

Claim: The optimal prediction coefficients $\{a_n\}_{n=1,\ldots,N}$ satisfy the system of equations given by

$$\sum_{n=1}^{N} a_n \text{csinc}((n-k)T) = \text{csinc}(kT), \tag{4}$$

for $k = 1, \ldots, N$. The prediction coefficients are independent of the prediction point t.

Proof. Consider the bandpass signals with Fourier representations as described above. Let set S be the union of the intervals $(-W_2, -W_1)$ and (W_1, W_2), and write $x(t) = \int_S X(f)e^{j2\pi ft}df$.

Optimality in the prediction coefficients means here that we minimize the error expression $\epsilon^2 := |x(t)-\sum_{n=1}^{N} a_n x(t-nT)|^2$. Using the integral representation above in the error expression ϵ^2 allows it to be bounded. That is, $\epsilon^2 \leq |\int_S X(f)e^{j2\pi ft}[1-\sum_{n=1}^{N} a_n e^{-j2\pi fnT}]df|^2$, which is bounded by $||x||_2^2 \cdot \epsilon_I$, where

$$\epsilon_I := \int_S |1-\sum_{n=1}^{N} a_n e^{-j2\pi fnT}|^2 df. \tag{5}$$

This error integral (5) is an $N-$dimensional function of the $\{a_i\}_{i=1,\ldots,N}$. Standard optimization methods lead to the conditions

$$\sum_{n=1}^{N} a_n \int_{W_1}^{W_2} \cos(2\pi fT(n-k))df = \int_{W_1}^{W_2} \cos(2\pi fTk)df, \tag{6}$$

for $k = 1, \ldots, N$. The integrals in (6) are similar. Evaluating the one on the right with k fixed finds that integral equal to $(\sin(2\pi TkW_2) - \sin(2\pi TkW_1))/(2\pi Tk)$. The use of trigonometric identities makes this equal to

$$\frac{\sin(\pi Tk(W_2-W_1))\cos(2\pi Tk((W_1+W_2)/2))}{\pi Tk}$$

which equals $B \cdot \text{csinc}(kT)$, where $B = W_2 - W_1$ is bandwidth and $f_c = (W_1+W_2)/2$ is the center frequency. Applying this same result to the integrals in the sum on the left in (6) leaves the left side of (6) equal to $B\sum_{n=1}^{N} a_n \text{csinc}((n-k)T)$. This shows that the system to solve for the prediction coefficients has the form (4) as claimed and completes the proof.

Note that the system matrix in (4) is a symmetric, Toeplitz matrix. In matrix form, the system includes the prediction coefficients which form the vector $[a_1, \ldots, a_N]$ of unknowns. The system has matrix

$$\begin{bmatrix} f(0) & f(T) & \ldots & f((N-1)T) \\ f(T) & f(0) & \ldots & f((N-2)T) \\ \vdots & \vdots & \ldots & \vdots \\ f((N-1)T) & f((N-2)T) & \ldots & f(0) \end{bmatrix}, \tag{7}$$

with $f(t) = \text{csinc}(t)$ as in (3). The right side of the matrix system (4) is the vector $b := [\text{csinc}(T), \ldots, \text{csinc}(NT)]$. Note that once B, f_c, and T are given, the solution of (4) determines the prediction coefficients for all bandpass signals with bandwidth B and center frequency f_c, sampled with sampling interval T. These three parameters are reduced to two in the determination of the prediction coefficients from (4). Since $\text{csinc}(kT) = \text{sinc}(BTk)\cos(2\pi f_c Tk)$, the two combination parameters are $\tau := BT$ and $\phi := f_c T$. These definitions give the two parameters that determine the set of corresponding prediction coefficients from (4). The entries in the matrix are thus given by

$$\text{csinc}(kT) = \text{sinc}(k\tau)\cos(2\pi k\phi). \tag{8}$$

The following table gives an example of the values of the prediction coefficients obtained from solving the system (4). The example is for the case when $\tau = 0.05$, which corresponds to a relatively small bandwidth. The table gives the values of the coefficients for a set of ϕ values, where this parameter relates to the center frequency of the passband.

Table 1. Bandpass Prediction coefficients, for $N = 5$ with $\tau = 0.05$, using the system solution method, for various values of ϕ.

ϕ	a_1	a_2	a_3	a_4	a_5
0.050	4.6964	-9.0655	8.9797	-4.5630	0.9518
0.075	4.3999	-8.1958	8.0245	-4.1254	0.8917
0.100	3.9950	-7.1000	6.8421	-3.5743	0.8096
0.125	3.4918	-5.8853	5.5501	-2.9634	0.7076
0.150	2.9026	-4.6706	4.2560	-2.3526	0.5882
0.175	2.2419	-3.5747	3.0381	-1.8016	0.4543
0.200	1.5260	-2.7051	1.9334	-1.3643	0.3092
0.225	0.7725	-2.1467	0.9350	-1.0836	0.1565
0.250	0.0000	-1.9543	0.0000	-0.9869	0.0000
0.275	0.7725	2.1467	0.9350	1.0836	0.1565
0.300	1.5260	2.7051	1.9334	1.3643	0.3092
0.325	2.2419	3.5747	3.0381	1.8016	0.4543

4. EXAMPLES OF PREDICTIONS FOR BANDPASS SIGNALS

The graphs in the following figures show a comparison between the above method of this paper and the standard LPC method for prediction. A speech signal was filtered to frequencies in the interval $W_1 = 0.5$ kHz to $W_2 = 1.0$ kHz. The speech signal was orginally sampled at 8kHz, and the parameters for (8) are then $\tau = (W_2 - W_1)/8 = 1/16$ and $\phi = f_c/8 = 3/32$. The LPC and the new method of this paper for bandpass signals were each applied for an entire interval of time values. In Figures 1 and 2, each method was successively applied to a set of $N = 12$ signal values to obtain a prediction, then that prediction value was plotted, and the pattern was repeated with the next set of 12 values. Both the original signal and the complete set of predictions are presented in the figures.

Figure 1. This graph shows the result of applying the LPC method to a signal with frequencies in the interval from 0.5 kHz to 1kHz. The number of samples was 12.

Figure 2. This graph shows predictions for the same signal as in Figure 1, but made with the methods of this paper. The number of samples was 12 for each prediction for each case.

As can be seen from these two figures, the methods of this paper are much more accurate than standard LPC methods. Using the methods of section 3, the prediction overlaps the signal in the figure 2 so that no error is apparent. Actual relative error is on the order of 10^{-5}, about four orders of magnitude better than the result from LPC.

Figures 3 and 4 show predictions for the same signal as for the previous figures, but with $N = 5$ samples in each case instead of $N = 12$. Similar to the previous figures, the methods of this paper are more accurate. Actual relative error is usually on the order of 10^{-3} using the methods of this paper, and that is about two orders of magnitude better than the predictions resulting from LPC.

Figure 3. Predictions made with the LPC method using $N = 5$ samples.

Figure 4. Predictions made with the methods of this paper using $N = 5$ samples.

The predictions made using the methods of this paper again overlap the signal in figure 4 so that no error is apparent. Actual error has magnitude on the order of a unit for a signal that has range of around a thousand units.

The next two figures (5 and 6) show the results of applying both the LPC method and the method of section 3 to a signal whose bandwidth is the same as in the figures above but whose center frequency is much higher. The bandwidth is still related to $\tau = 1/16 = 0.0625$ but the center frequency relates to $\phi = 0.39375$. For this high frequency signal, results are similar to the cases above, with the methods of this paper giving predictions that are so accurate that no difference is seen (in Figure 6) between the signal and the predictions.

Figure 5. Predictions made with the LPC method for a high-frequency signal using $N = 5$ samples.

Figure 6. Predictions made with the methods of this paper for a high frequency signal using $N = 5$ samples.

5. PROPERTIES OF THE PREDICTION COEFFICIENTS FOR BANDPASS SIGNALS

There is another approach to the computation of the prediction coefficients which involves the eigenvectors obtained from an extension of the matrix in (7). This is similar to methods that may be used to compute prediction coefficients for band-limited signals, [3]. Prediction coefficients may be found as in section 3 directly by numerically solving the matrix system (4), and that numerical solution can take advantage of the Toeplitz nature of the system. The eigenvector method, however, also provides accurate predictions and uses prediction coefficients with a number of special properties.

For this approach, let vector c have first component equal to 1, i.e. $c_0 = 1$, and let $c_n = -a_n$ for $n = 1, \ldots, N$, so that the second through $N+1$ components of c are the negative values of the prediction coefficients. Then the sum in the error expression ϵ_I from (5) can be written as $\sum_{n=0}^{N} c_n e^{-j2\pi f n T}$. With this notation, the error expression may be written as a quadratic form,

$$\epsilon_I = \sum_{n=0}^{N} \sum_{m=0}^{N} c_n c_m \int_S e^{j2\pi f(n-m)T} df. \qquad (9)$$

In particular, for a matrix A_2 with entries $A_{2m,n} = \int_S e^{j2\pi f(n-m)T} df$, the error expression ϵ_I is equal to the inner product given by

$$\epsilon_I = c^T A_2 c, \qquad (10)$$

where the superscript T stands for transpose. If c is an arbitrary vector, this shows that matrix A_2 is positive definite, since ϵ_I, see (5), is positive. The minimum value of the inner product (10) over the complete set of vectors of length $N+1$ and unit norm is known to be given by the minimum eigenvalue of the matrix. Because of the similarity of the form of the entries in A_2 compared to that in the matrix (7) as given above, it can be shown that the new matrix A_2 can be formed from that matrix A and the right side vector b of that system, by the relation expressed by the block matrix

$$A_2 = \begin{bmatrix} 1 & b^T \\ b & A \end{bmatrix}, \qquad (11)$$

where the superscript T here stands for the transpose of the vector. Because of the form of each of these block entries in A_2, the entries in this matrix may be given simply as an extension of the formula for the entries in A, so that

$$A_{2i,j} = \text{csinc}((i-j)T), \tag{12}$$

for $i, j = 0, \ldots, N$. Note that the values of the csinc function actually depend on combination parameters τ and ϕ as expressed in (8).

The eigenvector associated with the smallest eigenvalue of matrix A_2 can be used to find prediction coefficients which lead to small error in the prediction of bandpass signals. This particular vector c which relates to the prediction coefficients $\{a_n\}_{n=1,\ldots,N}$ has some restrictions, such as that $c_0 = 1$ as noted above. However, if we take an eigenvector of A_2 associated with the minimum eigenvalue, and normalize it so that its first component equals 1, then the remaining entries in that eigenvector lead to the prediction coefficients. Such coefficients will lead to a small value of ϵ_I, as described above. The representation (10), $\epsilon_I = c^T A_2 c$, can then be used in the error bound provided in the proof of (4) to find a bound on the error in the prediction.

Although the prediction coefficients obtained by solving the system (4) minimize the prediction error, we have found in numerical trials that predictions done using the eigenvector method as outlined above lead to comparable results. In many predictions, we have found that the peak signal values are predicted with slightly better accuracy using the prediction coefficients obtained from the eigenvector method.

For a comparison to the numerical values of the prediction coefficients obtained from (4), Table 2 presents values of the prediction coefficients obtained by using the appropriate eigenvector of matrix A_2 as outlined above.

Table 2. Bandpass prediction coefficients, for $N = 5$ and $\tau = 0.05$, using the eigenvector method.

ϕ	a_1	a_2	a_3	a_4	a_5
0.050	4.7446	-9.2473	9.2473	-4.7446	1.0000
0.075	4.5084	-8.5791	8.5791	-4.5084	1.0000
0.100	4.1858	-7.7119	7.7119	-4.1858	1.0000
0.125	3.7847	-6.7072	6.7072	-3.7847	1.0000
0.150	3.3150	-5.6337	5.6337	-3.3150	1.0000
0.175	2.7884	-4.5623	4.5623	-2.7884	1.0000
0.200	2.2178	-3.5591	3.5591	-2.2178	1.0000
0.225	1.6169	-2.6804	2.6804	-1.6169	1.0000
0.250	1.0002	-1.9678	1.9678	-1.0002	1.0000
0.275	1.6169	2.6804	2.6804	1.6169	1.0000
0.300	2.2178	3.5591	3.5591	2.2178	1.0000
0.325	2.7884	4.5623	4.5623	2.7884	1.0000
0.350	3.3150	5.6337	5.6337	3.3150	1.0000

Some of the properties of the values of the prediction coefficients may be observed by considering Table 2. For example, magnitudes of the coefficients for fixed ϕ have symmetry about the middle of the vector. Also, the set of coefficients has similarities about the $\phi = 0.25$ point. Compare, for example, the magnitudes of the coefficients for $\phi = 0.15$ with those associated with $\phi = 0.35$.

In 1978, Slepian, in [9], identified a set of vectors important to digital signal processing as the Discrete Prolate Spheroidal Sequences (DPSS). This set of vectors may be obtained as the set of eigenvectors of what we call the "sinc matrix" S for parameter τ whose entries are $S_{i,j} = \text{sinc}((i-j)\tau)$. We found [3] that an eigenvector associated with the smallest eigenvalue of this matrix could lead to prediction coefficients for band-limited signals. For bandpass signals, we consider the matrix A_2 above. As noted earlier (12), matrix A_2 in (11) is an $N+1 \times N+1$ matrix of entries that extends the form of the system matrix from (4) as expressed by (7). Because of its similarity to the sinc matrix, we call A_2 the "csinc matrix" since it has entries given by (12).

Because of their relation to the DPSS sequences, we call the eigenvectors of the csinc matrix A_2 the CDPSS, i.e. the cosine-modulated DPSS. The eigenvector associated with the smallest eigenvalue of the csinc matrix is used to obtain prediction coefficients as outlined above and as given in Table 2. Besides their use in the prediction coefficients computation, the complete set of eigenvectors share a number of properties with the DPSS.

Table 3. Complete set of CDPSS vectors, eigenvectors of the csinc matrix, for the case $N = 6, \tau = 1/8, \phi = 7/32$. Arranged from left to right as associated with the smallest to largest eigenvalue.

v_5	v_4	v_3	v_2	v_1	v_0
-.2134	.4437	-.2654	.6496	-.1801	.4824
.3652	.1121	.6501	.2702	.5419	.2546
-.5666	.5390	.0833	-.0705	.4171	-.4500
.5666	.5390	.0833	.0705	-.4171	-.4500
-.3652	.1121	.6501	-.2702	-.5419	.2546
.2134	.4437	-.2654	-.6496	.1801	.4824

The prediction coefficients are derived for this case from the eigenvector associated with the smallest eigenvalue as is discussed above, and that vector is labeled v_5 in Table 3. This labeling of the vectors follows the scheme of Slepian, who labeled v_0 as the eigenvector associated with the largest eigenvalue for the DPSS case. For windowing purposes, the vector v_0 is the one usually considered, but it is v_5 that is of interest for prediction here.

One of the properties of the eigenvectors is that each vector has a symmetry. This is the same for the CDPSS as it is for the standard DPSS. The symmetry may be seen in Table 3 above, and this property is that each vector is either symmetric about its middle or antisymmetric about its middle. We have an argument that provides a proof of this result, based on the set of CDPSS being the set of eigenvectors of a real, symmetric Toeplitz matrix. There is not room here to present that proof.

Another property of the CDPSS is a relation between vectors for "complementary" frequencies. This was described for the DPSS in [2]. Just as for the standard DPSS vectors, the complete set of unit norm eigenvectors of matrix A_2, see (12), for parameter τ are related to the corresponding set of vectors for parameter $\tau' = 1 - \tau$. If $b_n^{(i)}$ is the DPSS for the complementary parameter, then it is known that $a_n^{(i)} = (-1)^n b_n^{(N-1-i)}$, where N is the length of the vector and i is the ordering of the eigenvectors, where i goes from 0, the largest eigenvalue, to $N-1$, the smallest eigenvalue. We have a proof of this result, based on the trigonometric nature of entries of matrix A_2 for the set of CDPSS, but there is not room here to present that proof. For the csinc matrix and its eigenvectors, there is an additional complementary relation between eigenvectors related to parameters ϕ and $1/2 - \phi$, as well as a direct relation between those for ϕ and $1-\phi$. This complementary relation for ϕ may be seen from Table 2, as noted earlier.

The DPSS, [2], satisfy a three-term recurrence relation that is derived based on the construction of a tridiagonal matrix that commutes with the sinc matrix S, described above. There is a tridiagonal matrix [9] which shares the eigenvectors for a given sinc matrix, and is appropriate for very accurate calculation of the prediction coefficients for the band-limited case [3]. We have a new derivation of the results in [1] that show that no such tridiagonal commuting matrix can be constructed for the csinc matrix (12), except for the limiting "cosine matrix" with entries $\cos((i-j)\phi)$ obtained in the limit from the csinc matrix as parameter $\tau \rightarrow 0$. The limiting case also provides a set of coefficients for perfect predictions of single sinusoids, but there is insufficient space here to present these results. We remark that this result for the limit of the csinc matrix is similar to the limiting set of coefficients for the sinc matrix for band-limited functions. In that case, [3], the coefficients provide perfect reconstruction of polynomials of an appropriate degree, and the appropriate prediction coefficients are alternating binomial coefficients. This pattern can be seen in the leading row in Table 2 of $(4.7466, -9.2473, 9.2473, -4.7466, 1.0000)$ which are close to the alternating binomial coefficients of $(5, -10, 10, -5, 1)$. This occurs since both τ and ϕ are close to the limiting zero values in that row, so that that csinc matrix is close to the limiting case of the sinc matrix. The set of coefficients mentioned above that provides perfect prediction of single sinusoids is a set that is much smaller in magnitude than these binomial coefficients.

In closing, the authors are incorporating this work on bandpass signals as a part of a filter bank approach to linear prediction for band-limited signals. This involves a sub-band decomposition of the band-limited signal. Initial results have shown the filter bank approach capable of obtaining very accurate predictions.

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