1. Find a series solution about $x = 0$ for the initial value problem $y' + 2xy = 0, \quad y(0) = 5$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=1}^{\infty} a_n n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0 \qquad \text{Shift } n+1 \to n-1$$

$$\sum_{n=1}^{\infty} a_n x^{n-1} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n-1} = 0$$

- $n = 1 \implies a_1 = 0$
- $n \geq 2 \implies n a_n + 2a_{n-2} = 0 \implies a_n = \frac{-2}{n} a_{n-2}$

$a_2 = \frac{-1}{1} a_0$	$a_3 = 0$
$a_4 = \frac{-1}{2} a_2$	$a_5 = 0$
$a_6 = \frac{-1}{3} a_4$	$a_7 = 0$
$a_8 = \frac{-1}{4} a_6$	$a_9 = 0$
$\vdots$	$\vdots$
$a_{2k} = \frac{-1}{k} a_{2k-2}$	

$$\not{a}_2 \not{a}_4 \not{a}_6 \ldots a_{2k} = \frac{(-1)(-1)(-1)\ldots(-1)}{1 \cdot 2 \cdot 3 \cdot 4 \ldots k} a_0 \implies a_{2k} = \frac{(-1)^k}{k!} a_0.$$

So, $y = a_0 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} x^k a_0 \quad = \quad a_0 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} x^k \right)$

Using the initial conditions of $y(0) = 5 \implies a_0 = 5$.

Solution:

$$y = 5 \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} x^k \right)$$

2. Use the power series method about $x = 0$ to find 2 linearly independent solutions of $2y'' + xy' + y = 0$.

Let $y = \displaystyle\sum_{n=0}^{\infty} a_n x^n$

$$2\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 2a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^{n} + \sum_{n=0}^{\infty} a_n x^n = 0 \qquad \text{Shift } n \to n-2$$

$$\sum_{n=2}^{\infty} 2a_n n(n-1)x^{n-2} + \sum_{n=3}^{\infty} a_{n-2}(n-2) x^{n-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$\bullet n = 2 \implies 4a_2 + a_0 = 0 \implies a_2 = \dfrac{-1}{2\cdot 2}a_0$

$\bullet n \geq 3 \implies 2n(n-1)a_n + (n-2)a_{n-2} + a_{n-2} = 0 \implies a_n = \dfrac{-1}{2n}a_{n-2}$

$$\begin{array}{c|c} & a_3 = \dfrac{-1}{2\cdot 3}a_1 \\ a_4 = \dfrac{-1}{2\cdot 4}a_2 & a_5 = \dfrac{-1}{2\cdot 5}a_3 \\ a_6 = \dfrac{-1}{2\cdot 6}a_4 & a_7 = \dfrac{-1}{2\cdot 7}a_5 \\ a_8 = \dfrac{-1}{2\cdot 8}a_6 & a_9 = \dfrac{-1}{2\cdot 9}a_7 \\ \vdots & \vdots \\ a_{2k} = \dfrac{-1}{2\cdot (2k)}a_{2k-2} & a_{2k+1} = \dfrac{-1}{2\cdot (2k+1)}a_{2k-1} \end{array}$$

$$\not a_4 \not a_6 \not a_8 \dots a_{2k} = \frac{(-1)(-1)(-1)\dots(-1)}{2(4)\cdot 2(6)\cdot 2(8)\dots 2(2k)}a_2 = \frac{(-1)(-1)(-1)\dots(-1)}{2(4)\cdot 2(6)\cdot 2(8)\dots 2(2k)}\cdot\frac{-1}{2(2)}a_0$$

Here, $a_{2k} = \displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{2^k\cdot 2^k k!} = \sum_{k=2}^{\infty}\frac{(-1)^k}{2^{2k}k!}a_0$.

Similarly for the odds, $a_{2k+1} = \displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{2^k[1\cdot 3\cdot 5\cdot 7\dots(2k+1)]}a_1$.

Finally, $y_1 = a_0\left(1 - \dfrac{1}{4}x^2 + \displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{2^{2k}k!}x^{2k}\right)$

and $y_2 = a_1\left(x + \displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{2^k[1\cdot 3\cdot 5\cdot 7\dots(2k+1)]}x^{2k+1}\right)$.

3. Find a power series solution about $x = 0$ to the initial value problem
$(x^2 - 4) + 3xy' + y = 0, \quad y(0) = 4, y'(0) = 1.$

Let $y = \displaystyle\sum_{n=0}^{\infty} a_n x^n$

$$x^2 \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 4\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + 3x\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n - \sum_{n=2}^{\infty} 4a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} 3a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \qquad \text{Shift } n \to n-2$$

$$\sum_{n=4}^{\infty} a_{n-2}(n-2)(n-3)x^{n-2} - \sum_{n=2}^{\infty} 4a_n n(n-1)x^{n-2} + \sum_{n=3}^{\infty} 3a_{n-2}(n-2)x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0$$

- $n = 2 \implies -8a_2 + a_0 = 0 \implies a_2 = \dfrac{1}{2\cdot 4}a_0.$
- $n = 3 \implies (-24a_3 + 3a_1 + a_1)x = 0x \implies a_3 = \dfrac{1}{2\cdot 3}a_1.$
- $n \geq 4 \implies (n-2)(n-3)a_{n-2} - 4n(n-1)a_n + 3(n-2)a_{n-2} + a_{n-2} = 0$

$$\implies a_n = \frac{n-1}{4n}a_{n-2}$$

$$\begin{array}{r|l}
a_4 = \dfrac{3}{4\cdot 4}a_2 & a_5 = \dfrac{4}{4\cdot 5}a_3 \\
a_6 = \dfrac{5}{4\cdot 6}a_4 & a_7 = \dfrac{6}{4\cdot 7}a_5 \\
a_8 = \dfrac{7}{4\cdot 8}a_6 & a_9 = \dfrac{8}{4\cdot 9}a_7 \\
\vdots & \vdots \\
a_{2k} = \dfrac{2k-1}{4\cdot(2k)}a_{2k-2} & a_{2k+1} = \dfrac{2k}{4\cdot(2k+1)}a_{2k-1}
\end{array}$$

$$\begin{aligned}
\cancel{a_4}\ \cancel{a_6}\ \cancel{a_8}\ldots a_{2k} &= \frac{3\cdot 5\cdot 7\cdot 9\cdots(2k-1)}{4(4)\cdot 4(6)\cdot 4(8)\cdot 4(10)\cdots 4(2k)}a_2 \\
&= \frac{3\cdot 5\cdot 7\cdot 9\cdots(2k-1)}{4(4)\cdot 4(6)\cdot 4(8)\cdot 4(10)\cdots 4(2k)}\cdot\frac{1}{4\cdot 2}a_0 \\
&= \frac{3\cdot 5\cdot 7\cdots(2k-1)}{4^k 2^k k!}a_0 \\
&= \frac{3\cdot 5\cdot 7\cdots(2k-1)}{2^{3k}k!}a_0
\end{aligned}$$

So, $y_1 = a_0\left(1 + \dfrac{1}{8}x^2 + \displaystyle\sum_{k=2}^{\infty}\frac{3\cdot 5\cdot 7\cdots(2k-1)}{2^{3k}k!}x^{2k}\right)$

$$
\begin{aligned}
\cancel{a_5}\ \cancel{a_7}\ \cancel{a_9} \dots a_{2k+1} &= \frac{4 \cdot 6 \cdot 8 \cdot 10 \cdots (2k)}{4(5) \cdot 4(7) \cdot 4(9) \cdot 4(11) \cdots 4(2k+1)} a_3 \\
&= \frac{4 \cdot 6 \cdot 8 \cdot 10 \cdots (2k)}{4(5) \cdot 4(7) \cdot 4(9) \cdot 4(11) \cdots 4(2k+1)} \cdot \frac{2}{4 \cdot 3} a_1 \\
&= \frac{2^k k!}{4^k [3 \cdot 5 \cdot 7 \cdots (2k+1)]} a_1 \\
&= \frac{k!}{2^k [3 \cdot 5 \cdot 7 \cdots (2k+1)]} a_1
\end{aligned}
$$

So, $y_2 = a_1 \left(x + \frac{1}{6} x^3 + \sum_{k=2}^{\infty} \frac{k!}{2^k [3 \cdot 5 \cdot 7 \cdots (2k+1)]} x^{2k+1} \right)$

Imposing initial conditions, $y(0) = 4 \implies a_0 = 4$ and $y'(0) = 1 \implies a_1 = 1$.

<u>Solution:</u>

$$
y = 4 \left(1 + \frac{1}{8} x^2 + \sum_{k=2}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2k-1)}{2^{3k} k!} x^{2k} \right) + \left(x + \frac{1}{6} x^3 + \sum_{k=2}^{\infty} \frac{k!}{2^k [3 \cdot 5 \cdot 7 \cdots (2k+1)]} x^{2k+1} \right)
$$

4. Find a series solution about $x = 0$ for the differential equation $(1-x^2)y'' - 6xy' - 4y = 0$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - x^2 \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 6x \sum_{n=1}^{\infty} a_n n x^{n-1} - 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^{n} - \sum_{n=1}^{\infty} 6a_n n x^{n} - \sum_{n=0}^{\infty} 4a_n x^n = 0 \qquad \text{Shift } n \to n-2$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=4}^{\infty} a_{n-2}(n-2)(n-3)x^{n-2} - \sum_{n=3}^{\infty} 6a_{n-2}(n-2)x^{n-2} - \sum_{n=2}^{\infty} 4a_{n-2}x^{n-2} = 0$$

- $n = 2 \implies 2a_2 - 4a_0 = 0 \implies a_2 = 2a_0.$
- $n = 3 \implies (6a_3 - 6a_1 - 4a_1)x = 0x \implies a_3 = \frac{5}{3}a_1.$
- $n \geq 4 \implies n(n-1)a_n - (n-2)(n-3)a_{n-2} - 6(n-2)a_{n-2} - 4a_{n-2} = 0$

$$\implies a_n = \frac{n+2}{n} a_{n-2}$$

$a_4 = \frac{6}{4}a_2$	$a_5 = \frac{7}{5}a_3$
$a_6 = \frac{8}{6}a_4$	$a_7 = \frac{9}{7}a_5$
$a_8 = \frac{10}{8}a_6$	$a_9 = \frac{11}{9}a_7$
$\vdots$	$\vdots$
$a_{2k} = \frac{2k+2}{2k}a_{2k-2}$	$a_{2k+1} = \frac{2k+3}{2k+1}a_{2k-1}$

$$\begin{aligned} \cancel{a_4}\ \cancel{a_6}\ \cancel{a_8}\ldots a_{2k} &= \frac{\cancel{6}\cdot \cancel{8}\cdot \cancel{10}\cdots(2k+2)}{4\cdot \cancel{6}\cdot \cancel{8}\cdots \cancel{2k}}a_2 \\ &= \frac{2k+2}{4}\cdot 2a_0 \\ &= (k+1)a_0. \end{aligned}$$

So, $y_1 = a_0\left(1 + 2x^2 + \sum_{k=2}^{\infty}(k+1)x^{2k}\right)$.

$$
\begin{aligned}
\not{a}_5\ \not{a}_7\ \not{a}_9 \dots a_{2k+1} &= \frac{\not{7}\cdot\ \not{9}\cdot\ \not{1}1\cdots(2k+3)}{5\cdot\ \not{7}\cdot\ \not{9}\cdots(2k+1)}a_3 \\
&= \frac{2k+3}{5}\cdot\frac{5}{3}a_1 \\
&= \frac{2k+3}{3}a_1.
\end{aligned}
$$

So, $y_2 = a_1\left(x + \frac{5}{3}x^3 + \sum_{k=2}^{\infty}\frac{2k+3}{3}x^{2k+1}\right)$.

<u>General Solution:</u>

$$
y = a_0\left(1 + 2x^2 + \sum_{k=2}^{\infty}(k+1)x^{2k}\right) + a_1\left(x + \frac{5}{3}x^3 + \sum_{k=2}^{\infty}\frac{2k+3}{3}x^{2k+1}\right)
$$