Test Total

Name KEY

3450:335 Differential Equations Exam. II Spring 08
Clemons

$\bar{X} = 77$ $\quad$ min $= 21$
$\sigma = 18$ $\quad$ med $= 80$
$n = 38$ $\quad$ max $= 100$

1. Find the solution to $\begin{aligned} x' &= -x + 4y \\ y' &= -2x + 3y \end{aligned}$.

$$\dot{\underline{X}} = \begin{pmatrix} -1 & 4 \\ -2 & 3 \end{pmatrix} \underline{X}$$

$$\det \begin{pmatrix} -1-\lambda & 4 \\ -2 & 3-\lambda \end{pmatrix} = (-1-\lambda)(3-\lambda) + 8 = \lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4(5)}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i \quad 2$$

$$(A - (1+2i)I)\underline{X} = \begin{pmatrix} -1-(1+2i) & 4 \\ -2 & 3-(1+2i) \end{pmatrix} \underline{X}$$

$$= \begin{pmatrix} -2-2i & 4 \\ -2 & 2-2i \end{pmatrix} \underline{X} = 0$$

$$(-2-2i)x_1 + 4x_2 = 0 \implies x_2 = \left(\tfrac{1}{2} + \tfrac{1}{2}i\right)x_1$$

let $x_1 = 2$, $x_2 = 1 + i$ $\qquad \underline{X} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 6

$$\underline{X} = e^t \left(c_1 \left\{ \cos 2t \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin 2t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} + c_2 \left\{ \cos 2t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin 2t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \right)$$

4 $\qquad$ 4

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2. Solve the differential equation $y''' - 2y'' + y' = -12 + 6x + (8-4x)e^{-x}$.

$$m^3 - 2m^2 + m = m(m-1)^2 = 0$$

$$m = 0, 1, 1;\ 0, 0, -1, -1$$

$$y = \underbrace{c_1 + c_2 e^x + c_3 x e^x}_{y_h\ 3} + \underbrace{Ax + Bx^2 + ce^{-x} + dxe^{-x}}_{y_p}$$

$$y_p = Ax + Bx^2 + ce^{-x} + dxe^{-x}$$

$$y_p' = A + 2Bx - ce^{-x} + d(1-x)e^{-x}$$

$$y_p'' = 2B + ce^{-x} + d(-2+x)e^{-x}$$

$$y_p''' = -ce^{-x} + d(3-x)e^{-x}$$

$$(-ce^{-x} + d(3-x)e^{-x}) - 2(2B + ce^{-x} + d(-2+x)e^{-x}) + (A + 2Bx - ce^{-x} + d(1-x)e^{-x}) = -12 + 6x + (8-4x)e^{-x}$$

$$(-4B + A) + (2B)x + (-c + 3d - 2c + 4d - c + d)e^{-x} + (-d - 2d - d)xe^{-x} = -12 + 6x + (8-4x)e^{-x}$$

$A - 4B = -12$ $\quad A = 0$

$2B = 6$ $\quad B = 3$

$-4c + 8d = 8$ $\quad c = 0$ 2

$-4d = -4$ $\quad d = 1$

$$y = \underbrace{c_1 + c_2 e^x + c_3 x e^x + 3x^2 + xe^{-x}}_{1}$$

2 2 2

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3. Given that

$$y = x^4 + x^2 \text{ is a solution to } y'' - 4y' + 4y = 4x^4 - 12x^3 + 15x^2 + 6x$$

find the solution of $y'' - 4y' + 4y = -2x^4 + 6x^3 - \dfrac{15}{2}x^2 - 3x + 4$, subject to $y(0) = 2$ and $y'(0) = 3$.

$$y_P = \underbrace{-\tfrac{1}{2}(x^4+x^2)}_{4} + \underbrace{1}_{3}$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$y = \underbrace{c_1 e^{2x} + c_2 x e^{2x}}_{4} + -\tfrac{1}{2}(x^4+x^2) + 1$$

$$y' = 2c_1 e^{2x} + c_2(1+2x)e^{2x} - \tfrac{1}{2}(4x^3+2x)$$

$$y(0) = 2 = c_1 + 1 \quad \Rightarrow c_1 = 1$$

$$y'(0) = 3 = 2c_1 + c_2 \qquad c_2 = 1$$

$$y = \underbrace{\underbrace{e^{2x}}_{3} + \underbrace{x e^{2x}}_{3} + -\tfrac{1}{2}(x^4+x^2) + 1}_{1}$$

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4. Find the solution to $x^2y'' + xy' + 9y = -\tan(3\ln x)$.

$$m(m-1)+m+9$$
$$m^2+9=0$$
$$m=\pm 3i$$
$$y_h = C_1\cos(3\ln x) + C_2\sin(3\ln x) \qquad y_p = u_1\cos(3\ln x)+u_2\sin(3\ln x)$$

4

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

NOTE: To get f the eqn must be in stand form.

$$\begin{pmatrix} \cos(3\ln x) & \sin(3\ln x) \\ -\frac{3}{x}\sin(3\ln x) & \frac{3}{x}\cos(3\ln x) \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{x^2}\tan(3\ln x) \end{pmatrix}$$

2

3 $$u_1' = \frac{\begin{vmatrix} 0 & \sin(3\ln x) \\ -\frac{1}{x^2}\tan(3\ln x) & \frac{3}{x}\cos(3\ln x) \end{vmatrix}}{\begin{vmatrix} \cos(3\ln x) & \sin(3\ln x) \\ -\frac{3}{x}\sin(3\ln x) & \frac{3}{x}\cos(3\ln x) \end{vmatrix}} = \frac{\frac{1}{x^2}\sin(3\ln x)\tan(3\ln x)}{\frac{3}{x}} = \frac{1}{3x}\cdot\frac{\sin^2(3\ln x)}{3x\cos(3\ln x)} = \frac{1}{3x}\left(\sec(3\ln x)-\cos(3\ln x)\right)$$

2 $$u_1 = \int \frac{1}{3x}\left[\sec(3\ln x) - \cos(3\ln x)\right]dx$$
$$= \frac{1}{9}\left[\ln\left|\sec(3\ln x)+\tan(3\ln x)\right| - \sin(3\ln x)\right]$$

3 $$u_2' = \frac{\begin{vmatrix} \cos(3\ln x) & 0 \\ -\frac{3}{x}\sin(3\ln x) & -\frac{1}{x^2}\tan(3\ln x) \end{vmatrix}}{\frac{3}{x}} = \frac{-\frac{1}{x^2}\sin(3\ln x)}{\frac{3}{x}} = -\frac{1}{3x}\sin(3\ln x)$$

2 $$u_2 = \int -\frac{1}{3x}\sin(3\ln x) = \frac{1}{9}\cos(3\ln x)$$

$$y = y_h + y_p = y_h + u_1y_1 + u_2y_2$$
$$= C_1\cos(3\ln x) + C_2\sin(3\ln x) + \frac{1}{9}\ln\left|\sec(3\ln x)+\tan(3\ln x)\right|\cos(3\ln x)$$
$$+\frac{1}{9}\sin(3\ln x)\cos(3\ln x)$$

2

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5. Given $y_1 = e^{-5x}$ is a solution to $xy'' + (5x-1)y' - 5y = 0$, find a second linearly independent solution.

$$y'' + \left(5 - \frac{1}{x}\right)y' - \frac{5}{x}y = 0$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx \quad 4$$

$$= e^{-5x} \int \frac{e^{-\int 5 - \frac{1}{x}\,dx}}{e^{-10x}}\,dx$$

$$= e^{-5x} \int \frac{e^{-5x + \ln x}}{e^{-10x}}\,dx$$

$$= e^{-5x} \int x e^{5x}\,dx \quad 4$$

$$u = x \quad dv = e^{5x}\,dx$$

$$du = dx \quad v = \frac{1}{5}e^{5x}$$

$$= e^{-5x}\left(\frac{x}{5}e^{5x} - \int \frac{1}{5}e^{5x}\,dx\right) \quad 4$$

$$= e^{-5x}\left(\frac{x}{5}e^{5x} - \frac{1}{25}e^{5x}\right)$$

$$= \frac{x}{5} - \frac{1}{25} \quad 4$$

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6.

(a) A 10 lb weight stretches a spring 3 inches. This weight is removed from the spring and replaced by a mass of 1.6 slugs. This mass is released 4 inches above equilibrium with a downward velocity of 5/4 ft/sec. There is no dampening in the system. **Set-up but do not solve** the equation of motion and initial conditions.

$F = kS \qquad 10 = k \cdot \frac{1}{4} \quad k = 40$ (2)

$1.6\ddot{x} + 40x = 0$ (4)

$x(0) = -\frac{1}{3}$ (2)

$\dot{x}(0) = 5/4$

8 pts

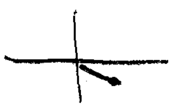
(b) The solution to the above is $x(t) = -\frac{1}{3}\cos(5t) + \frac{1}{4}\sin(5t)$. Find all times when the mass attains a displacement below the equilibrium position that is one-half the amplitude of the motion. Indicate which times correspond to upward motion and which to downward motion.

$x(t) = -\frac{1}{3}\cos 5t + \frac{1}{4}\sin 5t = A\sin(5t + \delta)$

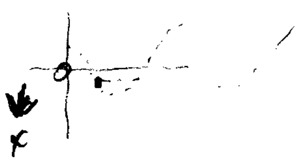
$A = \sqrt{\left(-\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2} = \sqrt{\frac{16+9}{3^2 \cdot 4^2}} = \frac{5}{12}$ (2)

$\sin\delta = -\frac{1}{3}$

$\cos\delta = \frac{1}{4}$



$\delta = \tan^{-1}\left(-\frac{4}{3}\right) = -.92729$ (2)



$\frac{5}{12}\sin(5t - .92729) = \frac{1}{2}\left(\frac{5}{12}\right)$

$5t - .92729 = \begin{cases} \frac{\pi}{6} + 2\pi k \\ \frac{5\pi}{6} + 2\pi k \end{cases} \quad k \in \mathbb{Z}$ (2)

$t = \frac{\left(\frac{\pi}{6} + .92729 + 2\pi k\right)}{5}$ down $k = 0, 1, 2, 3, \ldots$ (2)

$t = \frac{\frac{5\pi}{6} + .92729 + 2\pi k}{5}$ up $k: 0, 1, 2, 3, \ldots$

8 pts