Diff Eq - 3450:335
EXAM #2 S07

NAME Key
ROW

100 Points

Show **ALL** your work.

1. Find the general solution to $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 6e^{-2x} + 2x.$

$m = -2, 0, 0$

$y = y_c + y_p$

$y = e^{mx} \Rightarrow m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0$

$m = -2, -2$

$\Rightarrow y_c = C_1 e^{-2x} + C_2 x e^{-2x}$ (4)

m's $= -2, -2, -2, 0, 0$ Und coef.

$\Rightarrow y = C_1 e^{-2x} + C_2 x e^{-2x} + \underbrace{Ax^2 e^{-2x} + B + Cx}_{y_p}$ (3) (1) (2)

$y_p' = A[2xe^{-2x} - 2x^2 e^{-2x}] + C$

$y_p'' = A[2e^{-2x} - 8xe^{-2x} + 4x^2 e^{-2x}]$

$y'' + 4y' + 4y = 6e^{-2x} + 2x$

$A[2e^{-2x} - 8xe^{-2x} + \cancel{4x^2e^{-2x}}] + 4A[2xe^{-2x} - \cancel{2x^2e^{-2x}}] + 4C$

$+ \cancel{4Ax^2e^{-2x}} + 4B + 4Cx = 6e^{-2x} + 2x$

$xe^{-2x}: \; -8A + 8A = 0$ ok

$e^{-2x}: \; 2A = 6 \Rightarrow A = 3$ (4)

$x: \; 4C = 2 \Rightarrow C = \frac{1}{2}$

$1: \; 4C + 4B = 2 + 4B = 0 \Rightarrow B = -\frac{1}{2}$

$\Rightarrow y = C_1 e^{-2x} + C_2 x e^{-2x} + 3x^2 e^{-2x} - \frac{1}{2} + \frac{1}{2}x$ (2)

16 Points

$y'' + \frac{1}{x}y' + \frac{1}{x}\left(x - \frac{1}{4x}\right)y = 0$

2. Find the general solution to $xy'' + y' + \left(x - \frac{1}{4x}\right)y = 0$ if $\frac{\cos x}{\sqrt{x}}$ is one solution of this ODE.

$$y = \frac{\cos x}{\sqrt{x}} \int \frac{e^{-\int \frac{1}{x}dx}}{\frac{\cos^2 x}{x}}\,dx = \frac{\cos x}{\sqrt{x}} \int \frac{e^{-\ln x}}{\frac{\cos^2 x}{x}}\,dx = \frac{\cos x}{\sqrt{x}} \int \frac{\frac{1}{x}}{\frac{\cos^2 x}{x}}\,dx$$

2 2 4 2

$$= \frac{\cos x}{\sqrt{x}} \int \sec^2 x\,dx = \frac{\cos x}{\sqrt{x}} \cdot \tan x = \frac{\sin x}{\sqrt{x}}$$

2 2 2

16 Points

$$\text{so } y = C_1 \frac{\cos x}{\sqrt{x}} + C_2 \frac{\sin x}{\sqrt{x}}$$

4

3. A solution to $x^2y'' + 5xy' + 4y = 4\ln x + 16$ is $y = \ln x$. A solution to $x^2y'' + 5xy' + 4y = -16x^2 + 8$ is $y = -x^2 + 2$. Find the general solution to $x^2y'' + 5xy' + 4y = \ln x + 8x^2$.

$y = y_c + y_p$

16 Points

Get y_c Solve $x^2y'' + 5xy' + 4y = 0$

$y = x^m \Rightarrow m(m-1) + 5m + 4 = m^2 + 4m + 4 = 0 = (m+2)^2$

4 so $m = -2, -2$

$y_c = C_1 x^{-2} + C_2 x^{-2}\ln x$

Get y_p by superposition $a(4\ln x + 16) + b(-16x^2 + 8) = \ln x + 8x^2$

$a = 1/4 \quad b = -1/2$

$$\text{so } y = C_1 x^{-2} + C_2 x^{-2}\ln x + \frac{1}{4}\ln x - \frac{1}{2}(-x^2 + 2)$$

2 2 4 4

32 Points

4. Find the general solution to $x^2y'' - 4x^2y' + 4x^2y = e^{2x}$.

$$y'' - 4y' + 4y = \frac{e^{2x}}{x^2}$$

16 Points

$y = y_c + y_p$ Get $y_c \Rightarrow m^2 - 4m + 4 = (m-2)^2 = 0 \Rightarrow m = 2, 2$

So $y_c = C_1 e^{2x} + C_2 x e^{2x}$ (4)

Try $y_p = V_1 e^{2x} + V_2 x e^{2x}$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} \quad (2)$$

$$V_1' = \frac{-xe^{2x} \cdot \frac{e^{2x}}{x^2}}{e^{4x}} = \frac{-1}{x}$$

So $V_1 = \int \frac{-1}{x} dx = -\ln x$ (3)

$$V_2' = \frac{e^{2x} \frac{e^{2x}}{x^2}}{e^{4x}} = \frac{1}{x^2}$$

$V_2 = \int \frac{1}{x^2} dx = \frac{-1}{x}$ (3)

So $y = C_1 e^{2x} + C_2 x e^{2x} - (\ln x) e^{2x} - \frac{1}{x} x e^{2x}$ (4)

5. Solve the system of equations:
$$\frac{dx}{dt} = x - 4y$$
$$\frac{dy}{dt} = x + y.$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

16 Points

Get e-values $\begin{vmatrix} 1-\lambda & -4 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 1 + 4 = \lambda^2 - 2\lambda + 5 = 0$

$$\lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i \quad (4)$$

Get e-vector $\lambda = 1 + 2i$

$$\begin{pmatrix} 1-(1+2i) & -4 \\ 1 & 1-(1+2i) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow k_1 - 2ik_2 = 0$$

(2) so $k_1 = 2ik_2$

e-vector $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 2ik_2 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 + 2ik_2 \\ k_2 + 0i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (2) for $k_2 = 1$

$$\text{So } \begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^t \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right] \quad (4)$$
$$+ C_2 e^t \left[\begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t \right] \quad (4)$$

32 Points

6. A 32 lb weight is attached to a spring hanging from the ceiling. This causes the spring to stretch 8/9 ft on coming to rest at equilibrium. There is no damping in this system.

Initially the weight is released $\sqrt{3}$ ft above the equilibrium position with an upward velocity of 6 ft/sec.

6a. Write down the governing differential equation and initial conditions for the motion of the weight. DO NOT SOLVE THE EQUATION.

$m : \frac{32}{32} = 1 \qquad 32 = k \cdot \frac{8}{9} \Rightarrow k = \frac{32 \cdot 9}{8} = 36 \text{ lb/ft}$

$c = 0$

(2) (2)

$x'' + 36x = 0$

$x(0) = -\sqrt{3} \qquad x'(0) = -6$

(1) (1)

6 Points

6b. The answer to the above differential equation is $x(t) = -\sqrt{3}\cos(6t) - \sin(6t)$. Find the amplitude and phase shift, ϕ, of this motion.

$A = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = 2$ (2)

$\phi = \tan^{-1}\frac{-\sqrt{3}}{-1} = \tan^{-1}\sqrt{3} = \pi/3 + \pi = 4\pi/3$ (2)

3rd quad $\qquad = 4.189$

4 Points

6c. Find all of the times when the weight passes through the point $\sqrt{3}$ ft below the equilibrium point heading up. Do the same heading down.

$\sqrt{3} = 2\sin\left(6t + 4\pi/3\right)$

$6t + \frac{4\pi}{3} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} + 2n\pi, \quad \frac{2\pi}{3} + 2n\pi \qquad 0.524 + \frac{\pi}{3} = 0.5245$

(2) / (1) (2) (1)

so $t = \frac{\pi/3 - 4\pi/3}{6} + \frac{2n\pi}{6} = -\pi/6 + \frac{n\pi}{3} \quad n = 1, 2, \ldots$ down

$\frac{\frac{2\pi}{3} - 4\pi/3}{6} + \frac{2n\pi}{6} = -\frac{\pi}{9} + \frac{n\pi}{3} \quad n = 1, 2, \ldots$ up

$= -0.349 + \pi/3 = 1.396$

6 Points

6d. If a force of the form $f(t) = \cos(6t)$ lbs is applied to the system, describe what happens to the motion of the weight as $t \to \infty$. Hint: consider the form of x_p.

$x'' + 36x = \cos 6t$

$m = \pm 6i \qquad m = \pm 6i$ (2)

so $x(t) = c_1\cos 6t + c_2\sin 6t + At\cos 6t + Bt\sin 6t$

(2) Blow up as $t \to \infty$

resonance

4 Points

20 Points