$\bar{x} = 30$
$\sigma = 13.8$
$n = 33$
$\min = 48$
$\text{med} = 85$
$\max = 98$

Test Total

Name Key

3450:335 Differential Equations Exam. I
Clemons

Spring 2008

1. Consider $\dfrac{dy}{dx} + d\left[\dfrac{9x^2y^2 + 8x^3y - ax}{6x^3y + 2x^4 + by^2}\right] = 0$:

(a) Classify the equation as to whether it is *linear, exact, homogeneous or bernoulli* in the case:

(i) $d = 1$ $\quad (6x^3y + 2x^4 + by^2)dy + (9x^2y^2 + 8x^3y - ax)dx = 0$

$$18x^2y + 8x^3 = 18x^2y + 8x^3 \quad \underline{\text{exact}}.$$

(ii) $d = -1$ and $a = b = 0$

$$\frac{dy}{dx} = \frac{9x^2y^2 + 8x^3y}{6x^3y + 2x^4} \qquad \underline{\text{Homogeneous}}$$

5 pts

(b) Solve the problem in the case $d = 1$, $a = 2$ and $b = 3$.

$$(6x^3y + 2x^4 + 3y^2)dy + (9x^2y^2 + 8x^3y - 2x)dx = 0$$

$F_y = 6x^3y + 2x^4 + 3y^2 \qquad F_x = 9x^2y^2 + 8x^3y - 2x$

$F = 3x^3y^2 + 2x^4y + y^3 + g(x)$

$F_x = 9x^2y^2 + 8x^3y + g'(x) = 9x^2y^2 + 8x^3y - 2x$

$g'(x) = -2x$

$g = -x^2$

$F = C$

$$\underline{3x^3y^2 + 2x^4y + y^3 - x^2 = C}.$$

8 pts

2. Find the general solution to $e^x dy = (e^x y - \frac{1}{2}xy^3)dx$.

$$\frac{dy}{dx} - y = -\frac{1}{2}xe^{-x}y^3 \quad 3$$

Bernoulli

$$v = y^{1-3} = y^{-2}$$

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \quad 3$$

$$-2y^{-3}\frac{dy}{dx} - y(-2y^{-3}) = -\frac{1}{2}xe^{-x}y^3 \cdot -2y^{-3}$$

$$v' + 2v = xe^{-x} \quad 2$$

$$\text{I.F. } e^{\int 2dx} = e^{2x} \quad 2$$

$$(e^{2x}v)' = xe^{x}$$

$$e^{2x}v = (x-1)e^x + C \quad 2$$

$$v = (x-1)e^{-x} + Ce^{-2x} \quad 2$$

$$\frac{1}{y^2} = (x-1)e^{-x} + Ce^{-2x} \quad 1$$

$$y^2 = \frac{1}{(x-1)e^{x} + Ce^{-2x}}$$

15 pts

3. Find the general solution to $\frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x}$.

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\frac{y}{x} + 1\right)$$

$$u = \frac{y}{x} \quad \text{or} \quad \frac{dy}{dx} = u + x\frac{du}{dx} \quad 4$$

$$u + x\frac{du}{dx} = u(\ln u + 1) \quad 4$$

$$x\frac{du}{dx} = u\ln u + \cancel{u} - \cancel{u}$$

$$\frac{1}{u\ln u}\,du = \frac{1}{x}\,dx \quad 3$$

$$\ln(\ln u) = \ln x + C$$

$$\ln u = Cx \quad 2$$

$$u = e^{cx}$$

$$\frac{y}{x} = e^{c \cdot x}$$

$$\underline{y = xe^{c \cdot x}} \quad 2$$

15 pts

4. Solve $\frac{dy}{dx} = (-x + 4y + 2)^2$ by using the substitution $z = -x + 4y + 2$.

$$z = -x + 4y + 2$$

3 $$\frac{dz}{dx} = -1 + 4\frac{dy}{dx} \Rightarrow \frac{1}{4}\left(\frac{dz}{dx} + 1\right) = z^2$$

$$\frac{dz}{dx} = 4z^2 - 1$$

$$\frac{1}{4z^2-1}\,dz = dx \quad 3$$

$(2z-1)(2z+1)$

3 $$\frac{\frac{1}{2}}{2z-1} + \frac{-\frac{1}{2}}{2z+1} = x + C$$

2 $$\frac{1}{4}\ln(2z-1) - \frac{1}{4}\ln(2z+1) = x + C$$

$$\ln\sqrt[4]{\frac{2z-1}{2z+1}} = x + C$$

$$\sqrt[4]{\frac{2z-1}{2z+1}} = ce^{x}$$

$$\frac{2z-1}{2z+1} = ce^{4x}$$

$$2z - 1 = 2ce^{4x}\cdot z + ce^{4x}$$

$$2(1 - ce^{4x})z = ce^{4x} + 1$$

2 $$z = \frac{ce^{4x}+1}{2(1-ce^{4x})}$$

$$-x + 4y + 2 = \frac{ce^{4x}+1}{2(1-ce^{4x})}$$

2 $$y = -\frac{1}{2} + \frac{1}{4}x + \frac{ce^{4x}+1}{8(1-ce^{4x})}$$

15 pts

5. The population of fairy shrimp in an Ohio pool is known to decay at a rate proportional to the square root of (10 minus the population). Initially there are 10 fairy shrimp and after 1 day there are 6 fairy shrimp, what is the first time the population is zero?

3

$$\frac{dP}{dt} = k\sqrt{10-P}.$$

$P(0)=10$

$P(1)=6$ 2

$P(\tilde{t})=0.$

$$\frac{dP}{\sqrt{10-P}} = k\,dt$$

$$-2\sqrt{10-P} = kt+C$$

$$\sqrt{10-P} = -\frac{k}{2}t+C$$

$$10-P = \left(-\frac{k}{2}t+C\right)^2$$

4 $P = 10 - \left(-\frac{k}{2}t+C\right)^2$ $P(0)=10 \Rightarrow C=0$ 2

$$P(1) = 10 - \left(-\frac{k}{2}\right)^2 = 6$$

$$4 = \frac{k^2}{4} \Rightarrow k=4$$ 2

$$P(t) = 10 - 4t^2 = 0$$ 2

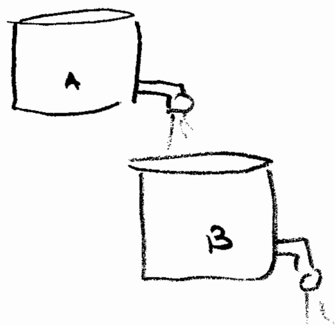
$t = \sqrt{10/4}$ days.

≈ 1.58 days

15 pts

6a. Two tanks are designed as shown in the figure. Tank 1 initially contains 800 gallons of water with 90 kg of sugar in solution. Water containg 6 kg of sugar per gallon is entering the tank at the rate of 7 gal/min. The well-stirred mixture is flowing out of the tank at the rate of 11 gal/min. If $A(t)$ denotes the amount of sugar in kg in the tank at any time t in minutes. **SET UP BUT DO NOT SOLVE** the differential equation(s) and condition(s) necessary to find $A(t)$.

7



$V(0) = 100$ gal w/ $A(0) = 90$ kg 1

$r_i = 7$ gal/min $\quad c_i = \frac{6}{1} \frac{kg}{g}$.

$r_o = 11$ gal/min

1 $\quad$ 1 $\quad$ 3

$$\frac{dA}{dt} = r_i c_i - r_o \cdot c_o = 7 \cdot 6 - 11 \cdot \frac{A(t)}{V(t)}$$

where $V(t) = 400 + (7-11)t = 800 - 4t$

Tank 1 is empty @ $t = 200$ min

6 pts

(b) Tank 2 initially contains 800 gallons of pure water. The well-stirred mixture is flowing out of the tank at the rate of 8 gal/min. If $B(t)$ denotes the amount of sugar in kg in the tank at any time t in minutes, **SET UP BUT DO NOT SOLVE** the differential equation(s) and condition(s) necessary to find $B(t)$.

1 $\quad \frac{dB}{dt} = r_i \cdot c_i - r_o c_o$

2 $\quad$ 2

$$= 11 \cdot \frac{A(t)}{800-4t} - 8 \cdot \frac{B(t)}{800+3t}$$

$V_2(t) = 800 + (11-8)t$

$= 800 + 3t$

$r_o = 8$ gal/min

$B(0) = 0$ 1

6 pts

7. A crime profiler investigating a ~~homicides~~ 90 minutes found a can ~~of~~ soda on the kitchen counter. The soda measured 50°F and the room temperature measured 70°F. The can was thought to originally be in the refigerator which measured 40°F. ~~an hour~~ later The Soda's Temperature was measured to be 60°

(a) **SET UP BUT DO NOT SOLVE** the differential equation and condition necessary to find the temperature $T(t)$ of the soda at any time t.

$T(0) = 50^{\circ}F \quad T_m = 70$ 3

$T(90) = 60^{\circ}F$

$$\frac{dT}{dt} = k(T - 70) \quad 2$$

5 pts

(b) The solution to your differential equation is $T(t) = 70 + (50 - 70)e^{-kt}$. Find k.

$$60 = 70 - 20e^{-k \cdot 90} \quad 2$$

$$\frac{1}{2} = e^{-k \cdot 90}$$

$$\frac{\ln 1/2}{-90} = k = .0077016$$

3

5 pts

(c) At what time was the soda taken out of the refrigerator?

$$40 = 70 + (50 - 70)e^{-k \cdot t} \quad 2$$

$$\frac{-30}{-20} = e^{-kt}$$

$$\frac{\ln 3/2}{-.007016} = t \quad 2$$

$52.65 \text{ min} = t$ 1

before The Soda's Temp was measured to be 50°F.

5 pts