Test Total

Name KEY

Sample Final S'08 Linear Algebra 3450:312 Dr. Clemons
Show your work.

1. Let $A = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$.

(a) Show that the characteristic equation is $(1-\lambda)^2(2-\lambda)$.

$$\begin{array}{c} + \\ - \\ + \end{array}\begin{vmatrix} 1-\lambda & -3 & 5 \\ 0 & 1-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ -2 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda)$$

Expand along 1st Column

10 pts

(b) Eigenvalue $\lambda = 2$ has eigenvector $\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$. Find a basis for the eigenspace of $\lambda = 1$.

$(A-I)\underline{v} = \begin{pmatrix} 0 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix}\underline{v} = \underline{0}$ $\qquad$ $\left.\begin{array}{l} v_2 = 5/3\, v_3 \\ v_2 = 1/2\, v_3 \end{array}\right\} \Rightarrow v_2 = v_3 = 0$

any e'vector is $\underline{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

10 pts

(c) Is matrix A diagonalizable? If so, write a matrix P and diagonal matrix D such that $P^{-1}AP = D$.

The eigenspace for $\lambda = 1$ has dim 1 $(g.m(\lambda=1) = 1)$, but the Algebraic multiplicity $(a.m(\lambda=1) = 2)$ is 2. A is not diagonalizable

5 pts

Page 1 Total (25)

2. Find the least squares solution to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$A^T A \hat{\underline{x}} = A^T \underline{b}$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \hat{\underline{x}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \hat{\underline{x}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 2 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2/3 & 1/3 \\ 2 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2/3 & 1/3 \\ 0 & 2/3 & 1/3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1/2 \end{array}\right)$

$\hat{x} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$ $\quad A\hat{\underline{x}} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$

10 points

3. Consider the transformation $T : \mathbf{R}^4 \to \mathbf{R}^2$, given by $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 7x_2 + x_3 \\ x_1 + x_3 \end{pmatrix}$

(a) Show that T is a linear transformation.

$T\left(\alpha\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = T\begin{pmatrix} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \\ \alpha x_3 + y_3 \end{pmatrix} = \begin{pmatrix} (\alpha x_1 + y_1) + 7(\alpha x_2 + y_2) + (\alpha x_3 + y_3) \\ (\alpha x_1 + y_1) + (\alpha x_3 + y_3) \end{pmatrix}$

$= \alpha\begin{pmatrix} x_1 + 7x_2 + x_3 \\ x_1 + x_3 \end{pmatrix} + \begin{pmatrix} y_1 + 7y_1 + y_3 \\ y_1 + y_3 \end{pmatrix} = \alpha T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + T\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

8 pts

(b) Find the matrix representation of T with respect to the standard bases.

$T\underline{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\underline{e}_1 + 1e_2$

$T\underline{e}_2 = \begin{pmatrix} 7 \\ 0 \end{pmatrix} = 7\underline{e}_1 + 0e_2$ $\quad A = \begin{pmatrix} 1 & 7 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$T\underline{e}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\underline{e}_1 + 1e_2$

10 pts

Page 2 Total (28)

4. Determine whether or not the vectors 1, x^2, and $x^2 - 2$ are linearly independent in $\mathbf{P}_2$. Do these vectors form a basis for $\mathbf{P}_2$?

NOTE: $2(1) + 1(x^2-2) = x^2$
so the vectors are lin dep

alt. Solve
$\alpha_1(1) + \alpha_2(x^2) + \alpha_3(x^2-2) = 0$
to find $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

The set is not a basis since it is lin. dep. moreover
P_2 would require 3 lin. indep. vectors

10 pts

5. Show that the null space for matrix $A_{m \times n}$ is a subspace? Is it a subspace of $\mathbf{R}_m$ or $\mathbf{R}_n$

1. $\underline{0} \in \text{Null } A$ since $A\underline{0} = \underline{0}$

2. Let $\underline{u}, \underline{v} \in \text{Null } A$ i.e. $A\underline{u} = A\underline{v} = \underline{0}$,

The $A(\alpha\underline{u} + \underline{v}) = \alpha A\underline{u} + A\underline{v} = \alpha\underline{0} + \underline{0} = 0$. So $\alpha\underline{u} + \underline{v} \in \text{Null } A$.

The set is non-empty & is closed under scalar mult & add'n.

7 pts

6. If λ is an eigenvalue of matrix A having corrsponding eigenvector $\mathbf{x}$, what is and eigenvalue and eigenvector for A^2.

$A\underline{x} = \lambda\underline{x}$

$A^2\underline{x} = A(A\underline{x}) = A(\lambda\underline{x}) = \lambda A\underline{x} = \lambda(\lambda\underline{x}) = \lambda^2\underline{x}$.

6 pts

Page 3 Total (22)

7. Let W be the subspace of $\mathbf{R}^4$ spanned by the linearly independent vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 3 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ 2 \\ 3 \end{pmatrix}$. The Gram-Schmidt orthonormalization process has produced $\mathbf{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$.

(a) Find $\mathbf{u}_3$

$$\underline{v}_3' = \underline{v}_3 - (\underline{v}_3 \cdot \underline{u}_1)\underline{u}_1 - (\underline{v}_3 \cdot \underline{u}_2)\underline{u}_2$$

$$= \begin{pmatrix} 5 \\ 0 \\ 2 \\ 3 \end{pmatrix} - \left[\begin{pmatrix} 5 \\ 0 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}\right] \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} - \left[\begin{pmatrix} 5 \\ 0 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}\right] \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{pmatrix} - \begin{pmatrix} 3/2 \\ -3/2 \\ -3/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\underline{u}_3 = \frac{\underline{v}_3'}{\|\underline{v}_3'\|} = \frac{\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}}{2} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

10 pts

(b) Given $\mathbf{y} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$, find the projection of $\mathbf{y}$ onto W.

$$\mathrm{Pr}_W \underline{y} = [\underline{y} \cdot \underline{u}_1]\underline{u}_1 + [\underline{y} \cdot \underline{u}_2]\underline{u}_2 + \underbrace{[\underline{y} \cdot \underline{u}_3]}_{\to 0}\underline{u}_3$$

Since $\underline{y}$ is a lin comb of $\underline{u}_1$ & $\underline{u}_2$

In fact for this reason

$\underline{y} = \mathrm{Pr}_W \underline{y}$

$$= \left[\begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}\right] \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \left[\begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}\right] \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix} + \left[\begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}\right] \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \underline{0}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

i.e. $\underline{y} \in W$

10 pts

Page 4 Total (20)

8. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix}$.

(a) Find an upper triangular matrix R such that $A = QR$.

$$QR = A \Rightarrow R = Q^T A$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

10 pts

(b) Use the QR factorization to find the least-squares solution of $A\mathrm{x} = \mathbf{b}$.

$$A \hat{\underline{x}} = \hat{\underline{b}}$$

$$QR\,\underline{x} = \underline{b} \Rightarrow R\hat{\underline{x}} = Q^T \underline{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \hat{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

<u>NOTE</u>: $A\hat{x} = \begin{pmatrix} 1 & 1 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix} = \underline{b} = \begin{pmatrix} 1 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Evidently, $\underline{b} \in \mathrm{Col}\,A$.

10 pts

Page 5 Total (20)

9. Given that $T(1,0,1) = (2,3)$ and $T(0,1,1) = (4,-2)$,

6 pts

(a) Define T as a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$.

T is completely determined by its action on a basis. add $\begin{pmatrix}0\\0\\1\end{pmatrix}$ to $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$

& define $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \leftarrow$ any vector from $\mathbb{R}^2$.

(b) Find $\mathbf{x}$ in $\mathbf{R}^3$ such that $T(\mathbf{x}) = (0,-8)$

NOTE: $-2(2,3)+1(4,-2) = (0,-8)$

Hence, $T\left(-2\begin{pmatrix}1\\0\\1\end{pmatrix} + 1\begin{pmatrix}0\\1\\1\end{pmatrix}\right) = -2T\begin{pmatrix}1\\0\\1\end{pmatrix} + T\begin{pmatrix}0\\1\\1\end{pmatrix} = -2\begin{pmatrix}2\\3\end{pmatrix} + 1\begin{pmatrix}4\\-2\end{pmatrix} = \begin{pmatrix}0\\-8\end{pmatrix}$

Choose $\underline{x} = -2\begin{pmatrix}1\\0\\1\end{pmatrix} + 1\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\1\\-1\end{pmatrix}$.

10 pts

10. Let $A = \begin{pmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$.

(a) Write the solution set for the system $A\mathbf{x} = \mathbf{b}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ -2 & 5 & 4 & 3 \\ -5 & 6 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 9 & 18 & 9 \\ 0 & 16 & 32 & 16 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + 3x_3 = 1$

$x_2 + 2x_3 = 1$

x_3 Free

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 - 3x_3 \\ 1 - 2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \underline{x_3}$$

10 pts

(b) What is the rank and nullity of A?

nullity(A) = 1 $\leftarrow$ 1 free variable

Columns A = 3. By rank-nullity Thm.

Rank(A) + 1 = 3

or

Rank(A) = 2.

6 pts

Page 6 Total (32)