

Name KEY
 Sample Test 1 Linear Algebra 3450:312

Show your work.

1. Use row-reduction on the *augmented matrix* to find all solutions to the following linear system. Write your solution in *vector form*.

$$\begin{aligned} x_1 - 2x_2 + x_4 &= 2 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 7 \\ 2x_1 - 4x_2 + 2x_4 &= 4 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 2 & -4 & 1 & 1 & 7 \\ 2 & -4 & 0 & 2 & 4 \end{array} \right) \begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3 \end{array} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{aligned} x_3 &= 3 + x_4 \\ x_1 &= 2 + 2x_2 - x_4 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

15 points

2. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find the *LU*-decomposition of A . Find A^{-1} using inverse formula for a 2×2 matrix.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow[E_1]{-3R_1 + R_3 \rightarrow R_3} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = U$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \quad \text{so } E_1 \cdot A = U \text{ \& } A = E_1^{-1} U$$

$$E_1^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = L, \quad \text{Hence } A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} = \frac{1}{4-6} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

15 points

3. Suppose that $\{\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4\}$ is linearly dependent.

Is it necessarily true that we can always write \underline{u}_4 as a linear combination of $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$?

Carefully explain your answer.

NO Suppos $\underline{u}_1 = \underline{0}$, $\underline{u}_2 = -\underline{u}_3$, $\underline{u}_4 \neq \alpha \underline{u}_2$

for any α .

If $\underline{u}_4 = \alpha_1 \cdot \underline{0} + \alpha_2 \underline{u}_2 + \alpha_3 (-\underline{u}_2)$

Then $\underline{u}_4 = (\alpha_2 - \alpha_3) \underline{u}_2$ ~~✗~~.

10 points

4. Use row-reduction to find the inverse of $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 2 & 3 \end{bmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{l} -R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array} \right) \begin{array}{l} -\frac{1}{2}R_3 \rightarrow R_3 \\ -\frac{1}{2}R_2 \rightarrow R_2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array} \right)$$

$$\begin{array}{l} -R_2 + R_1 \rightarrow R_1 \\ -\frac{1}{2}R_3 + R_1 \rightarrow R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array} \right) \begin{array}{l} \frac{1}{2}R_3 + R_2 \rightarrow R_2 \\ -\frac{5}{2}R_3 + R_1 \rightarrow R_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -9/2 & 1/2 & 5/2 \\ 3/2 & -1/2 & -1/2 \\ 2 & 0 & -1 \end{pmatrix}$$

15 points

5. Show that if $\{x_1, x_2, x_3\}$ is linearly dependent and $A_{3 \times 3}$ is *one-to-one*, then $\{y_1, y_2, y_3\}$ is linearly dependent, where $y_1 = Ax_1$, $y_2 = Ax_2$ and $y_3 = Ax_3$.

$$\text{Consider } \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = 0$$

$$\text{Then } \alpha_1 Ax_1 + \alpha_2 Ax_2 + \alpha_3 Ax_3 = 0$$

$$\text{OR } A(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3) = 0$$

$$\text{Since } A \text{ is 1-1, } \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0.$$

$$\text{lin. indep of } \{x_1, x_2, x_3\}$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

10 points

6. Find the values of a, b, c so that the following matrix multiplication holds :

$$\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ -\frac{2}{3} & -\frac{4}{3} & \frac{4}{3} \\ 1 & \frac{5}{2} & 1 \end{bmatrix}$$

$$\begin{pmatrix} 2 & 1 & 2 \\ 2a & a-1 & 2a+2 \\ 2b & b-c & 2b+2c+4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ -\frac{2}{3} & -\frac{4}{3} & \frac{4}{3} \\ 1 & \frac{5}{2} & 1 \end{pmatrix}$$

$$2a = -\frac{2}{3} \Rightarrow a = -\frac{1}{3}$$

$$2b = 1 \Rightarrow b = \frac{1}{2}$$

$$b+c = \frac{5}{2} \Rightarrow c = \frac{5}{2} - b = \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2$$

10 points

7. Suppose that $A \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $A\underline{x} = \underline{0}$ has exactly one free variable.

Find the general solution to $A\underline{x} = \underline{0}$.

$$A \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

10 points

8. Suppose that T is a linear transformation from \mathbb{R}^2 to \mathbb{R}^4 , which satisfies

$$T(2, 1) = (1, 3, 5, 7) \text{ and } T(4, 7) = (-1, 1, -1, 1)$$

Evaluate $T(3, 5)$

$$(3, 5) = \alpha(2, 1) + \beta(4, 7)$$

$$\begin{aligned} 2\alpha + 4\beta &= 3 \\ \alpha + 7\beta &= 5 \end{aligned}$$

$$\begin{aligned} 10 - 14\beta + 4\beta &= 3 \Rightarrow \beta = \frac{7}{10} \\ \alpha &= 5 - 7\beta \\ \alpha &= 5 - 49 \cdot \frac{1}{10} = \frac{1}{10} \end{aligned}$$

$$\begin{aligned} T(3, 5) &= T\left(\frac{1}{10}(2, 1) + \frac{7}{10}(4, 7)\right) = \frac{1}{10}T(2, 1) + \frac{7}{10}T(4, 7) \\ &= \frac{1}{10}(1, 3, 5, 7) + \frac{7}{10}(-1, 1, -1, 1) \\ &= \left(\frac{-6}{10}, 1, \frac{-2}{10}, \frac{14}{10}\right) \end{aligned}$$

10 points

9. A is a 3x3 matrix. To reduce it to the *identity matrix*, we first exchange row_1 and row_2 , and then replace row_3 by $row_3 + row_2$.

What is A ?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = I$$

5 points

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} I = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} = A$$