Avg 86.2

Test Total

Name KEY

Test 1 Linear Algebra 3450:312 Dr. Clemons Spring 2008
Show your work.

1. Use row-reduction on the *augmented matrix* to find all solutions to the following linear system. Write your solution in *vector form.*

$$x_1 - 2x_2 + x_4 = 2$$
$$2x_1 - 5x_2 + x_3 + x_4 = 7$$
$$2x_1 - 4x_2 + 2x_4 = 4$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 2 & -5 & 1 & 1 & 7 \\ 2 & -4 & 0 & 2 & 4 \end{array}\right) \xrightarrow[-2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & -1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-R_2 \to R_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{2R_2+R_1 \to R_1} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 3 & -4 \\ 0 & 1 & -1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad (9)$$

$$\begin{aligned} x_1 &= -4 + 2x_3 - 3x_4 \\ x_2 &= 3 + x_3 - x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad (6)$$

15 points

2. Prove that if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent, then so are $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2, 2\mathbf{v}_3\}$.

$$0 = \alpha_1(\underline{v}_1 + \underline{v}_2) + \alpha_2(\underline{v}_1 - \underline{v}_2) + \alpha_3(2\underline{v}_3) \quad 5$$
$$= (\alpha_1+\alpha_2)\underline{v}_1 + (\alpha_1-\alpha_2)\underline{v}_2 + (2\alpha_3)\underline{v}_3 = 0$$

Since $\underline{v}_1, \underline{v}_2$ & $\underline{v}_3$ are lin. indep. (3)

$$\alpha_1+\alpha_2 = 0 \qquad 2\alpha_1 = 0 \Rightarrow \alpha_1 = 0$$
$$\alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2 \qquad \Downarrow \; \alpha_2 = 0 \quad (2)$$
$$2\alpha_3 = 0 \Rightarrow \alpha_3 = 0$$

So $\{\underline{v}_1+\underline{v}_2, \underline{v}_1-\underline{v}_2, 2\underline{v}_3\}$ is lin indep

10 points

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3. Let $A = \begin{pmatrix} 6 & 9 \\ 4 & 5 \end{pmatrix}$.

(a) Find the the LU-decomposition of A , i.e. $A = LU$, where L is a lower diagonal matrix and U is an upper diagonal matrix.

$$\begin{pmatrix}6&9\\4&5\end{pmatrix} \xrightarrow[E_1 = \begin{pmatrix}1/6&0\\0&1\end{pmatrix}]{\frac{1}{6}R_1 \to R_1} \begin{pmatrix}1&3/2\\4&5\end{pmatrix} \xrightarrow[E_2 = \begin{pmatrix}1&0\\4&1\end{pmatrix}]{-4R_1+R_2 \to R_2} \boxed{\begin{pmatrix}1&3/2\\0&-1\end{pmatrix} = U}$$

4

2 each

So $E_2 E_1 A = U$ i.e. $\boxed{\begin{pmatrix}1&0\\4&1\end{pmatrix}\begin{pmatrix}1/6&0\\0&1\end{pmatrix}} A = U$

2 each

$$A = \begin{pmatrix}1/6&0\\0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0\\4&1\end{pmatrix}^{-1} U = \boxed{\begin{pmatrix}6&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\4&1\end{pmatrix}} U$$

$$\begin{pmatrix}1&0\\2/3&1\end{pmatrix}\begin{pmatrix}6&9\\0&-1\end{pmatrix}$$

$$= \begin{pmatrix}6&0\\4&1\end{pmatrix}\begin{pmatrix}1&3/2\\0&-1\end{pmatrix} = LU$$

3

$$= \begin{pmatrix}6&9\\4&5\end{pmatrix} = A.$$

$$L = \begin{pmatrix}6&0\\4&1\end{pmatrix} \quad \boxed{U = \begin{pmatrix}1&3/2\\0&-1\end{pmatrix}}$$

15 points

(b) Using the adjoint formula find A^{-1}

2 each

$$A^{-1} = \frac{1}{6\cdot5-9\cdot4}\begin{pmatrix}5&-9\\-4&6\end{pmatrix} = -\frac{1}{6}\begin{pmatrix}5&-9\\-4&6\end{pmatrix} = \begin{pmatrix}-5/6&3/2\\2/3&-1\end{pmatrix}$$

2

Check: $A^{-1}A = \begin{pmatrix}-5/6&3/2\\2/3&-1\end{pmatrix}\begin{pmatrix}6&9\\4&5\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I_2.$

10 points

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4. Show that if $A_{3\times 3}$ is *one-to-one and onto* and $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linearly independent then $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ are linearly ~~dependent~~ independent, where $\mathbf{y}_1 = A\mathbf{x}_1$, $\mathbf{y}_2 = A\mathbf{x}_2$ and $\mathbf{y}_3 = A\mathbf{x}_3$.

Consider $\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = 0$ (3)

$\Rightarrow \alpha_1 A\underline{x}_1 + \alpha_2 A \underline{x}_2 + \alpha_3 A \underline{x}_3 = 0$

$\Rightarrow A(\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \alpha_3 \underline{x}_3) = 0$ (3)

A 1-1 $\Rightarrow A\underline{x} = \underline{0}$ has only trivial soln

Hence $\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \alpha_3 \underline{x}_3 = \underline{0}$. (2)

By lin indep of $\underline{x}_1, \underline{x}_2, \underline{x}_3$

(2)

$\alpha_1 = \alpha_2 = \alpha_3 = 0$ & y_1, y_2, y_3 are lin indep

10 points

5. Use row-reduction to find the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[(2)]{R_1+R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[(3)]{-\frac{1}{2}R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1/2 \end{array}\right) \xrightarrow[(4)]{\text{Swap } R_2 \,\&\, R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[(3)]{-R_2+R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 1 & 1 & 1/2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1/2 \\ 1 & 1 & 1/2 \end{pmatrix} \quad (2)$$

15 points

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6. Suppose that $A\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\-1\\2\end{pmatrix} = A\begin{pmatrix}2\\1\\1\end{pmatrix}$.

(a) Is $A : \mathbf{R}^3 \to \mathbf{R}^3$ 1-1? Explain?

NO, By defn it is not.

2

(b) Is A invertible? Explain?

NO, invertibility means 1-1 & onto

2

(c) Does $A\mathbf{x} = \mathbf{0}$ have only the trivial solution? If not, find a solution to $A\mathbf{x} = 0$.

No, equivalent of 1-1 means $A\underline{x}=\underline{0}$ has a nontrivial Soln.

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$$A\begin{pmatrix}1\\2\\3\end{pmatrix} - A\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\2\end{pmatrix} - \begin{pmatrix}0\\-1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\parallel$$

$A\begin{pmatrix}-1\\1\\2\end{pmatrix}$. Let $\underline{x} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$

10 points

7. Suppose that T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, which satisfies $T(1,2) = (3,1,2)$ and $T(4,5) = (-2,4,5)$

Evaluate $T(2,1)$.

$(2,1) = \alpha(1,2) + B(4,5)$

$-2(\ \alpha + 4B = 2\ )$

$2\alpha + 5B = 1$

$+$

$-3B = -3$ or $B = 1$. Hence, $\alpha = 2 - 4B = -2$

$(2,1) = -2(1,2) + 1(4,5)$ 5

$T(2,1) = T(-2(1,2) + 1\ (4,5))$

$= -2T(1,2) + 1\cdot T(4,5)$ 5

$= -2\begin{pmatrix}3\\1\\2\end{pmatrix} + 1\begin{pmatrix}-2\\4\\5\end{pmatrix} = \begin{pmatrix}-8\\2\\1\end{pmatrix}$ 5

15 points

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