Show all your work.

1. For the function \( z = x^2 y^2 \)
   a) Calculate the normal vector to this surface at the point \( (x, y, z) = (1, 3, 9) \).

   \[
   \nabla f = \langle 2 - 2x^2 y, 2x^2 y^2 \rangle
   \]

   \[
   \nabla f = \langle -2, 6 \rangle
   \] (3 points)

   b) Find the equation of the tangent plane at the point \( (1, 3, 9) \).

   \[
   \langle 2, 6 \rangle \cdot \langle x-1, y-3, z-9 \rangle = 0
   \]

   \[
   2(x-1) + 6(y-3) + 9 = 0
   \] (4 points)

   c) Find the direction of maximal increase of the function at the point \( (x, y) = (1, 3) \).

   \[
   \nabla f = \langle 4x, 6y \rangle
   \]

   \[
   \nabla f = \langle 8, 18 \rangle
   \] (4 points)

   d) Find the directional derivative of \( f(x, y) \) in the direction of \(-4i + 3j\) at the point \( (x, y) = (1, 3) \).

   \[
   D_{\hat{v}} f = \nabla f \cdot \frac{\langle -4, 3 \rangle}{\sqrt{16+9}}
   \]

   \[
   2 \cdot \langle 8, 18 \rangle = \frac{16 + 54}{5}
   \]

   \[
   = 18
   \] (6 points)

   \[
   \frac{18}{5} = -18
   \] (18 points)
2. Find the domain and range of \( z = \sqrt{x^2 + y^2 - 4} \), and sketch and label two level curves of this function.

\[
\text{Domain: } \{(x, y) \mid \sqrt{x^2 + y^2 - 4} \leq y\} \quad \text{Range: } \{z \geq 0\}
\]

\[
k = \sqrt{x^2 + y^2 - 4}
\]

\[
x^2 + y^2 = k^2 + 4
\]

\[
\text{Graphically: radius greater than or equal to 2.}
\]

8 Points

3. Use Lagrange multipliers to find the extrema of \( f(x, y, z) = 6x + 4y + 12z \) subject to the constraint \( x^2 + 2y^2 + 3z^2 = \frac{23}{4} \).

\[
W = 6x + 4y + 12z - \lambda \left( x^2 + 2y^2 + 3z^2 - \frac{23}{4} \right)
\]

\[
\begin{align*}
\Delta x &= 0 \\
\Delta y &= 2y - \lambda x = 0 \\
\Delta z &= 12z - \lambda z = 0 \\
W &= 6x + 4y + 12z - \lambda \left( x^2 + 2y^2 + 3z^2 - \frac{23}{4} \right)
\end{align*}
\]

\[
\frac{9}{\lambda} + \frac{2y}{x} + \frac{21}{z} - \lambda = \frac{23}{4}
\]

\[
\lambda = \pm 2
\]

10 Points

4. Given \( z = f(x, y) \) with \( x = g(t) \) and \( y = h(t) \), find \( \frac{dz}{dt} \) when \( t = 3 \), if \( g(3) = 2 \), \( g'(3) = 5 \), \( h(3) = 7 \), \( h'(3) = -4 \), \( f_x(2, 7) = 9 \), \( f_y(2, 7) = 6 \), \( f_x(2, 7) = 3 \) and \( f_y(2, 7) = -8 \).

\[
\frac{dz}{dt} = \frac{df}{dt} = f_x \cdot g'(t) + f_y \cdot h'(t)
\]

\[
= 9 \cdot 3 + 6 \cdot -8 = -3
\]

8 Points

26 Points
5. Find and then use the Second Partial Test to classify the critical points of 
\( f(x, y) = x^2 - 2xy + \frac{1}{3}y^3 - 3y. \)

\[
\begin{align*}
(1) & \quad f_x = 2x - 2y = 0 \implies x = y \quad \text{and} \\
(2) & \quad f_y = -2x + y^2 = 0 \implies y^2 = 2x \implies y = \sqrt{2x} \\
\text{Critical points:} & \quad (1, 1) \quad \text{and} \quad (1, -1)
\end{align*}
\]

\( f_{xx} = 2 > 0 \)

\( f_{yy} = 2 \)

Correlation:

\[
D = f_{xx}f_{yy} - (f_{xy})^2 = 2 - (2) = 0 \quad \text{is indeterminate at} \quad (1, 1)
\]

\[
D = 2 \cdot 2 - (-2)^2 = 8 > 0 \quad \text{at} \quad f_{xx} > 0 \quad \text{so} \quad \text{we have a local minimum at} \quad (1, 1)
\]

6. The dimensions of a closed rectangular box are measured as 80 cm, 60 cm and 50 cm, respectively with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

\[
SA = 2xy + 2xz + 2yz
\]

\[
\begin{align*}
\delta SA & = 2 \delta x + 2 \delta z + 2 \delta y \\
\delta A & = (27 + 2x + 2(x + z)) \delta x + 2(x + y) \delta y \\
& = 2(0.0 + 0.0) 0.2 + 2(0) 0.2 + 2(0 + 0) 0.2 \\
& = 0.2 + 0.2 + 0.2 + 0.2 \\
& = 0.82 \text{ cm}^2
\end{align*}
\]
1. Evaluate \[ \int_0^1 \int_y^2 e^x \, dx \, dy. \]

\[ \int_0^2 \int_y^2 e^x \, dx \, dy = \int_0^2 [e^2 - e^y] \, dy = e^2 - 2. \]

2. SET UP BUT DO NOT EVALUATE the integral in polar coordinates that represents the surface area of the portion of the sphere \( x^2 + y^2 + z^2 = 9 \) which lies inside the cylinder \( x^2 + y^2 = 3x \) and above the xy plane.

\[ S_A = \iint_R \sqrt{1 + \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right)} \, dA = \int_0^{\frac{3}{2}} \int_0^{\pi/3} (1 + \left( \frac{\sin^2 \theta}{3} + \frac{\sin^2 \theta}{3} \right)) \, r \, dr \, d\theta. \]
3. SET UP BUT DO NOT EVALUATE an integral representing the volume of the region bounded between \( z = 2x^2 + 2y^2 \) and \( z = 36 - 2x^2 - 2y^2 \)
   a) in cartesian coordinates.
   \[
   V = \iiint_{R} (36 - 2x^2 - 2y^2 - (2x^2 + 2y^2)) \, dx \, dy \, dz
   \]
   \[
   = \iiint_{R} 36 - 4x^2 - 4y^2 \, dx \, dy \, dz
   \]
   \[
   = \iiint_{R} (36 - 2) \, dx \, dy \, dz
   \]
   \[
   = \iiint_{R} 34 \, dx \, dy \, dz
   \]
   \[
   \Rightarrow V = \int_{0}^{2\pi} \int_{0}^{\sqrt{9 - z^2}} \left( \int_{-\sqrt{9 - z^2}}^{\sqrt{9 - z^2}} 34 \, dx \right) \, r \, dr \, dz
   \]
   \[
   = \int_{0}^{2\pi} \int_{0}^{\sqrt{9 - z^2}} \left( 68 \sqrt{9 - z^2} \right) \, r \, dr \, dz
   \]
   \[
   = \int_{0}^{2\pi} \left( 34 \pi \right) \, dz
   \]
   \[
   = 34\pi \int_{0}^{2\pi} \, dz
   \]
   \[
   = 34\pi [z]_{0}^{2\pi}
   \]
   \[
   = 34\pi[2\pi]
   \]
   \[
   = 68\pi
   \]
   b) in polar coordinates.
   \[
   \iiint_{R} (36 - 2r^2) \, r \, dr \, d\theta \, dz
   \]
   \[
   = \int_{0}^{2\pi} \int_{0}^{\sqrt{9 - z^2}} \left( \int_{0}^{\sqrt{9 - z^2}} 36 - 2r^2 \, r \, dr \right) \, dz
   \]
   \[
   = \int_{0}^{2\pi} \left( \int_{0}^{\sqrt{9 - z^2}} 36r - 2r^3 \, dr \right) \, dz
   \]
   \[
   = \int_{0}^{2\pi} \left( 18r^2 - \frac{1}{2}r^4 \right)_{0}^{\sqrt{9 - z^2}} \, dz
   \]
   \[
   = \int_{0}^{2\pi} \left( 18(9 - z^2) - \frac{1}{2}(9 - z^2)^2 \right) \, dz
   \]
   \[
   = \int_{0}^{2\pi} \left( 18\pi \right) \, dz
   \]
   \[
   = 18\pi \int_{0}^{2\pi} \, dz
   \]
   \[
   = 18\pi (2\pi)
   \]
   \[
   = 36\pi
   
4. SET UP BUT DO NOT EVALUATE an integral to find the volume of the solid in the first octant bounded by \( x = 0, \ y = 0, \ z = 0, \ x + y + z = 5 \) and inside \( x^2 + y^2 = 4 \).
   \[
   V = \iiint_{A} (5 - x - y) \, dx \, dy \, dz
   \]
   \[
   = \int_{0}^{\pi/2} \int_{0}^{\sqrt{4 - x^2}} \left( \int_{0}^{5 - x - y} (5 - x - y) \, dy \right) \, dx
   \]
   \[
   = \int_{0}^{\pi/2} \left( \int_{0}^{5 - x} (5 - x - y) \, dy \right) \, dx
   \]
   \[
   = \int_{0}^{\pi/2} \left( 5x - x^2 - \frac{1}{2}y^2 \right)_{0}^{5 - x} \, dx
   \]
   \[
   = \int_{0}^{\pi/2} \left( 5x - x^2 - \frac{1}{2}(5 - x)^2 \right) \, dx
   \]
   \[
   = \int_{0}^{\pi/2} \left( 5x - x^2 - \frac{25}{2} + \frac{5x}{2} \right) \, dx
   \]
   \[
   = \int_{0}^{\pi/2} \left( \frac{15x}{2} - \frac{3x^2}{2} - \frac{25}{2} \right) \, dx
   \]
   \[
   = \left( \frac{15x^2}{4} - \frac{x^3}{2} - \frac{25x}{2} \right)_{0}^{\pi/2}
   \]
   \[
   = \frac{15\pi^2}{8} - \frac{\pi^3}{8} - 25\pi
   \]
   \[
   = \frac{15\pi^2 - \pi^3 - 200\pi}{8}
   
   6 Points
   
18 Points