Vertex subgroups and vertex pairs in solvable groups

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Abstract. Green introduced the notion of a vertex of an indecomposable module $M$ of a finite group $G$ over an algebraically closed field of characteristic $p$. In this survey, we examine how the vertex subgroups, and related vertex pairs, arise and are used in the study of the representations of solvable groups. In particular, we show how vertices and their generalizations apply to the study of lifts of Brauer characters in solvable groups.

1. Introduction

For any finite group $G$, the representation theory of $G$ over the complex numbers (or any other “large enough” field of characteristic zero) is in some sense well understood. For instance, Frobenius reciprocity relates the behavior of induced representations, and their characters, to the restriction of the representations of $G$ to subgroups of $G$. However, many of the nice results over $\mathbb{C}$, such as Frobenius reciprocity, fail to carry over to the representation theory of finite groups over an algebraically closed field $k$ of prime characteristic $p$. In 1958, Green [14] introduced the notion of the vertex of an indecomposable $kG$-module $M$, which is a conjugacy class of $p$-subgroups that in some sense describe the behavior of $M$ with respect to induction from proper subgroups.

As the theory developed, interest arose in the interplay between the characteristic zero representations of a finite group $G$ and the representations over a field of prime characteristic $p$. For instance, the Fong-Swan theorem (see [27], for instance) showed that if $G$ is $p$-solvable, every irreducible Brauer character of $G$ can be lifted to an ordinary irreducible character of $G$. In the 1970’s, Isaacs developed a proof (see [18] and [19]) of the Fong-Swan theorem that made use of certain canonically defined characters of the $p$-solvable group $G$. Gajendragadkar [13] extended some of Isaacs’ results to $\pi$-separable groups, where $\pi$ is a set of primes. In the 1980’s, Isaacs [16], building on Gajendragadkar’s results, developed a “character theoretic” proof of the Fong-Swan theorem which extended the Fong-Swan theorem to $\pi$-separable groups, where irreducible $\pi$-partial characters (defined below) play the role of irreducible Brauer characters.

Interestingly, in the course of proving this later $\pi$-version of the Fong-Swan theorem, objects that behave like vertices of simple $kG$-modules appeared (see [16], and for a later and more explicit connection to vertices, see [30] or [21]).
Only now, a pair \((Q, \delta)\) was associated to each ordinary irreducible character of the \(\pi\)-separable group \(G\), where \(Q\) is a \(\pi'\)-subgroup of \(G\) and \(\delta\) is an ordinary irreducible character of \(Q\) (In the classical case, \(\pi' = \{p\}\)). These “vertex pairs” share many properties with the vertices defined by Green, and have shown to be quite useful in many problems in the representation theory of solvable groups. Our goal in these notes is to highlight some of the recent developments regarding vertex pairs of solvable groups and to see how they relate to the classical theory first developed by Green.

To this end, this paper is organized as follows: The next section briefly reviews Green’s notion of a vertex of an indecomposable \(kG\)-module, where \(k\) is an algebraically closed field of prime characteristic \(p\). In Section 3 we will see some of the properties of vertex subgroups that have recently been developed, including the first of several uniqueness results and some consequences, and a connection to the Alperin weight conjecture for solvable groups. In Section 4, we move to ordinary irreducible characters, and examine several different ways to produce vertex pairs for ordinary irreducible characters and properties that the vertex pairs have. Section 5 revisits the uniqueness question, and shows that with certain hypotheses, the vertex pairs have the same uniqueness properties that the vertex subgroups have. Section 6 examines the relationship between the vertex pairs and lifts of Brauer characters, and shows how vertex pairs can be applied to the study of lifts of Brauer characters. In Section 7 we discuss some recent results concerning the behavior of vertices with respect to normal subgroups. Section 8 is a discussion of Brauer characters of solvable groups with cyclic vertex subgroups, and we end with some open questions.

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\section{The classical notion of a vertex}

In this section we briefly review the notion of a vertex (see [14]) of an indecomposable \(kG\)-module \(M\), where \(G\) is a finite group and \(k\) is an algebraically closed field of prime characteristic \(p\). There is of course much that can be said about vertices in this context, and [1] and [32], for example, are excellent general references. Here we will mostly focus on the definition and the uniqueness property, as these are the results that will be emulated later in other contexts.

For the remainder of this section, \(k\) is an algebraically closed field of prime characteristic \(p\). If \(H\) is a subgroup of \(G\), \(L\) is a \(kH\)-module, and \(M\) is a \(kG\)-module, we will denote by \(M_H\) the restriction of \(M\) to \(H\) and by \(L^G\) the \(kG\)-module obtained by inducing \(L\) to \(G\). Unlike in the characteristic zero case, here it is not necessarily true that if \(L\) is a submodule of \(M_H\), then \(M\) is a submodule of \(L^G\). For instance, suppose \(H = 1\) and \(L\) is the trivial module of \(H\). If \(M\) is any module of \(G\), then \(L\) is a component of \(M_H\). However, the indecomposable modules that are components of \(L^G \cong kG\) in this case are exactly the projective indecomposable modules of \(kG\). Thus there is no way to “induce” \(M\) from \(H\) if \(M\) is not projective.

However, if we assume that \(H\) contains a full Sylow \(p\)-subgroup of \(G\), then we get a different result. If \(M\) is an indecomposable \(kG\)-module, then there is necessarily a \(kH\)-module \(L\) such that \(L\) is a component of \(L^G\). For a subgroup \(H\) of \(G\), we say a \(kG\)-module \(M\) is \(H\)-projective if there exists a \(kH\)-module \(L\) such that \(M\) is a component of \(L^G\). In his seminal work [14], Green proved the following:
role of the irreducible Brauer characters of \( \pi \)-separable groups or a \( \pi \)-group or a \( \pi \)-p-group. We begin with the definition of the set \( I_\pi(G) \) of \( \pi \)-separable groups. Define the subset \( G^o \subseteq G \) to be the set of all elements of \( G \) whose order is divisible only by primes in \( \pi \), and for a class function \( \alpha : G \to \mathbb{C} \), define \( \alpha^o \) to be the restriction of \( \alpha \) to \( G^o \). In the following definition, let \( \text{Char}(G) \) denote the set of ordinary characters of \( G \).

**Definition 3.1.** Let \( G \) be a \( \pi \)-separable group. Define the set \( I_\pi(G) \) of irreducible \( \pi \)-partial characters of \( G \) by

\[
I_\pi(G) = \{ \chi^o \mid \chi \in \text{Irr}(G) \text{ and } \chi^o \neq \alpha^o + \beta^o \text{ for any } \alpha, \beta \in \text{Char}(G) \}.
\]

It can be shown [20] that the set \( I_\pi(G) \) forms a basis for the set of class functions on \( G^o \), and in fact that if \( \chi \in \text{Irr}(G) \), then \( \chi^o \) is a nonnegative integer linear combination of the elements of \( I_\pi(G) \). Note that it follows immediately from the definition that the Fong-Swan property holds for \( I_\pi(G) \), by which we mean that if \( \varphi \in I_\pi(G) \), there necessarily exists a character \( \chi \in \text{Irr}(G) \) such that \( \chi^o = \varphi \). Moreover, note that the usual definition of character induction applies to the set \( I_\pi(G) \). Finally, we point out that in the case that \( \pi = \{p'\} \), the set \( I_\pi(G) \) is precisely \( I\text{Br}_p(G) \).

We would like to extend the notion of a vertex subgroup to the set \( I_\pi(G) \). In particular, we would like to associate to each irreducible \( \pi \)-partial character \( \varphi \in I_\pi(G) \) a \( \pi' \)-subgroup \( Q \) that has properties like those discussed in the previous section. Following Isaacs and Navarro [21], we have the following definition:

**Definition 3.2.** Let \( G \) be a \( \pi \)-separable group, and let \( \varphi \in I_\pi(G) \). Suppose there exists a subgroup \( H \subseteq G \) and a character \( \alpha \in I_\pi(G) \) such that \( \alpha^G = \varphi \) and such that \( \alpha \) has \( \pi \)-degree. If \( Q \) is a Hall \( \pi' \)-subgroup of \( H \), then we say \( Q \) is a vertex subgroup of \( \varphi \).
We will soon discuss the proof in [21] that such a subgroup \( Q \) exists and is unique up to conjugacy. First, however, we will show that if \( \pi = \{ p' \} \), so that \( I_\pi(G) = \text{IBr}_p(G) \), then the above definition for the vertex of an irreducible \( \pi \)-partial character is equivalent to the definition given in Section 2 for the corresponding simple module. Thus suppose that \( G \) is \( p \)-solvable and \( \varphi \in \text{IBr}_p(G) \). Let \( Q \) be the vertex subgroup defined in Definition 3.2, and suppose \( Q_1 \) is the vertex subgroup (as defined in Section 2) for the corresponding module. Since induced modules correspond to induced Brauer characters, and \( \alpha^G = \varphi \), then the \( kG \)-module corresponding to \( \varphi \) is induced from a module of the subgroup \( H \), and thus, replacing \( Q_1 \) by a conjugate if necessary, we see that the \( p \)-subgroup \( Q_1 \) of \( H \) is contained in (some conjugate of) the Sylow \( p \)-subgroup \( Q \) of \( H \). It follows immediately from the above definition that \( \varphi(1)_p = |P : Q| \) for every Sylow \( p \)-subgroup of \( G \) containing \( Q \). However, as mentioned in Section 2, we see that \( |P : Q_1| \) divides \( \varphi(1)_p \), and thus \( Q = Q_1 \).

3.2. Existence and uniqueness. Of course, we have not yet shown that if \( \varphi \in I_\pi(G) \) for some \( \pi \)-separable group \( G \), then a vertex subgroup for \( \varphi \) exists, and we have not shown that it is unique. The existence was originally shown by Huppert [15] in the context of Brauer characters. However, we will discuss the approach of Isaacs and Navarro in [21] that demonstrates the existence and uniqueness of the vertex subgroup in the setting of irreducible \( \pi \)-partial characters.

Suppose \( G \) is \( \pi \)-separable and that \( \varphi \in I_\pi(G) \). If \( \varphi(1) \) is a \( \pi \)-number, then every Hall \( \pi' \)-subgroup of \( G \) is a vertex, and thus the existence and uniqueness (up to conjugacy) of the vertex subgroup is a consequence of the basic results about Hall \( \pi' \)-subgroups of \( \pi \)-separable groups.

Now assume that \( \varphi(1) \) is not a \( \pi \)-number. We will associate to \( \varphi \) a pair \( (H, \alpha) \), where \( H \subseteq G \) and \( \alpha \in I_\pi(H) \) is such that \( \alpha^G = \varphi \). After first developing the basic Clifford theory of irreducible \( \pi \)-partial characters, it can easily be shown that if \( M_1 \) and \( M_2 \) are normal subgroups of \( G \) such that the irreducible constituents of \( \varphi_{M_1} \) and \( \varphi_{M_2} \) have \( \pi \)-degree, then the irreducible constituents of \( \varphi_{M_1M_2} \) have \( \pi \)-degree. Thus there is a unique normal subgroup \( N \) of \( G \) maximal with the property that the constituents of \( \varphi_N \) have \( \pi \)-degree, and necessarily \( N < G \). If \( \theta \in I_\pi(N) \) is a constituent of \( \varphi_N \), then it is not hard to see that the stabilizer \( G_\theta \) is proper in \( G \), and thus we let \( \psi \in I_\pi(G_\theta|\theta) \) be the Clifford correspondent for \( \varphi \). If \( \psi \) has \( \pi \)-degree, then we set \( (H, \alpha) = (G_\theta, \psi) \). If \( \psi \) does not have \( \pi \)-degree, we iterate this process, until we have a subgroup \( H \) of \( G \) and a character \( \alpha \in I_\pi(H) \) of \( \pi \)-degree such that \( \alpha^G = \varphi \). Thus we let \( Q \) be a Hall \( \pi' \)-subgroup of \( H \), and we have shown the existence of a vertex subgroup. We shall call the pair \( (H, \alpha) \) that we have just constructed the normal nucleus, or Navarro nucleus, of \( \varphi \in I_\pi(G) \), and we mention that later we will define a similar notion of the normal nucleus for characters \( \chi \in \text{Irr}(G) \).

We now briefly trace the argument of Isaacs and Navarro in [21] that shows that the vertex subgroup \( Q \) of \( \varphi \in I_\pi(G) \) is unique up to conjugacy. Note that in the construction of the vertex subgroup from the normal nucleus in the last paragraph, one could choose different constituents of \( \varphi_N \) and thus different Clifford correspondents at each step, but since these correspondents are conjugate via elements of \( G \), then it is clear that any two vertex subgroups constructed from normal nuclei of \( \varphi \) are conjugate.
However, we are not done, as there could conceivably be vertex subgroups for \( \varphi \) that are not constructed from normal nuclei. We need to show that given any subgroup \( U \) of \( G \) such that there exists an irreducible \( \pi \)-partial character \( \psi \in \Irr(U) \) of \( \pi \)-degree such that \( \psi^G = \varphi \), then every Hall \( \pi' \)-subgroup \( Q_{1} \) of \( U \) is conjugate to a vertex \( Q \) constructed via some normal nucleus. The key lemma in this argument shows that if the pair \((U, \psi)\) is as above, and if \( N \) is a normal subgroup of \( G \) such that the constituents of \( \varphi_N \) have \( \pi \)-degree, then the index \([UN : U] = |N : N \cap U|\) is a \( \pi \)-number, and thus \(|U|_{\pi'} = |UN|_{\pi'}\) and the pair \((U, \psi)\) may be replaced by the pair \((UN, \psi^{UN})\). By applying this lemma to the normal subgroup \( N \) that is maximal with the property that the constituents of \( \varphi_N \) have \( \pi \)-degree, Isaacs and Navarro use a series of reductions to show that the vertex subgroup \( Q_{1} \) is conjugate to a vertex subgroup constructed via a normal nucleus. Thus we have the following:

**Theorem 3.3.** [21] Suppose \( G \) is a \( \pi \)-separable group and \( \varphi \in \Irr(G) \). Then there exists a vertex subgroup \( Q \) for \( \varphi \), and any two vertex subgroups of \( \varphi \) are conjugate in \( G \).

### 3.3. Alperin weights.

In later sections of this paper we will discuss further properties of vertex subgroups and their generalizations to ordinary irreducible characters of \( \pi \)-separable groups. Before concluding this section, however, we feel obligated to give one justification as to why vertex subgroups are useful, and to do this we discuss the Alperin weight conjecture.

Let \( G \) be any finite group and \( p \) a prime. Define a *weight* of \( G \) to be a pair \((P, \varphi)\), where \( P \) is a \( p \)-subgroup of \( G \) and \( \varphi \in \Irr(N_{G}(P)/P) \) is such that \( \varphi(1)_{p} = |N_{G}(P)/P|_{p} \). It is clear that \( G \) acts by conjugation on the set of weights. The Alperin weight conjecture proposes that \(|\IBr(G)|\) is equal to the number of orbits of the action of \( G \) on the weights of \( G \). This conjecture is known to be true for symmetric groups, general linear groups, and various families of simple groups. Isaacs and Navarro stated and proved a refined version of the Alperin weight conjecture for \( \pi \)-separable groups in [21], and we now briefly discuss their result. (For more discussion of the history of the proof of the Alperin weight conjecture for solvable groups, see [21]).

Isaacs and Navarro refine and prove the Alperin weight conjecture using the vertex subgroup. We say a \( \pi \)-weight of a \( \pi \)-separable group is a pair \((Q, \varphi)\), where \( Q \) is a \( \pi' \)-subgroup of \( G \) and \( \varphi \in \Irr(N_{G}(Q)/Q) \) is such that \( \varphi(1)_{\pi'} = |N_{G}(Q)/Q|_{\pi'} \). For any \( \pi' \)-subgroup \( Q \) of a \( \pi \)-separable group \( G \), let \( I_{\pi}(G|Q) \) denote the set of characters in \( I_{\pi}(G) \) that have vertex subgroup \( Q \), and let \( w(Q) \) denote the set of \( \pi' \)-weights of \( G \) with first component \( Q \). Then we have the following refined version of the Alperin weight conjecture in [21]:

**Theorem 3.4.** Let \( G \) be a \( \pi \)-separable group and suppose that a Hall \( \pi' \)-subgroup of \( G \) is nilpotent. Then for every \( \pi' \)-subgroup \( Q \) of \( G \), we have
\[
|I_{\pi}(G|Q)| = |w(Q)|.
\]

Notice that the assumption that a Hall \( \pi' \)-complement of \( G \) is nilpotent is trivially satisfied if \( \pi = \{p'\} \). We also mention here that the “vertex” version of the Alperin weight conjecture given above fails if \( G \) is not solvable. Thus the Alperin weight conjecture (if true) is even more surprising, as it asserts that a correspondence exists that is somehow not natural.

One then recovers the \( \pi \)-version of the Alperin weight conjecture by summing over all conjugacy classes of \( \pi' \)-subgroups of \( G \).
4. Vertices of ordinary characters in solvable groups

4.1. $\pi$-special and $\pi$-factorable characters. We will have more to say about vertex subgroups of irreducible $\pi$-partial characters in Sections 6 and 7. Now, however, we turn our attention to developing the notion of a vertex pair for an ordinary irreducible character $\chi$ of a $\pi$-separable group $G$. There are several similar yet nonequivalent ways to define this notion. However, all of them involve the notion of a $\pi$-factorable character, which we now discuss.

We begin with the notion of a $\pi$-special character $\alpha$ of a $\pi$-separable group, originally due to Gajendragadkar in [13]. For more details of the results about $\pi$-special characters discussed below, see [26].

**Definition 4.1.** Suppose that $G$ is a $\pi$-separable group and $\alpha \in \text{Irr}(G)$. We say that $\alpha$ is $\pi$-special if

1. $\alpha(1)$ is a $\pi$-number, and
2. for each subnormal subgroup $S$ of $G$, and each irreducible constituent $\gamma$ of $\alpha|_S$, the order of the linear character $\text{det}(\gamma)$ is a $\pi$-number.

Of course, every character of a $\pi$-group is $\pi$-special, and one can intuitively think of $\pi$-special characters as characters of a $\pi$-separable group $G$ that behave like characters of a $\pi$-group. It is immediate from the definition that if $T$ is a subnormal subgroup of $G$, and $\alpha \in \text{Irr}(G)$ is $\pi$-special, then the constituents of $\alpha|_T$ are $\pi$-special. Also, $\pi$-special characters behave well with respect to induction from normal subgroups. In particular, if $N \triangleleft G$ has $\pi$-index, and $\gamma \in \text{Irr}(N)$ is $\pi$-special, then every constituent of $\gamma^G$ is $\pi$-special. If $N \triangleleft G$ has $\pi'$-index, then it need not be the case that any of the constituents of $\gamma^G$ are $\pi$-special, and in fact it turns out that in this case there exists a $\pi$-special character $\alpha$ of $G$ lying over $\gamma$ if and only if $\gamma$ is invariant in $G$, and in this situation, $\alpha \in \text{Irr}(G)$ is an extension of $\gamma$ and is the unique $\pi$-special character of $G$ lying over $\gamma$.

We now describe part of the relationship between the $\pi$-special characters and the irreducible $\pi$-partial characters. If $\alpha \in \text{Irr}(G)$ is $\pi$-special, then $\alpha|_H \in \text{Irr}(H)$ for every Hall $\pi$-subgroup $H \subseteq G$. (This fits with the intuitive notion that $\pi$-special characters of $G$ behave like characters of $\pi$-groups, and thus do not “see” the $\pi'$-elements of $G$.) Thus in particular, $\alpha' \in \text{Irr}_\pi(G)$, and it can be shown that the map $\alpha \rightarrow \alpha'$ is a bijection from the set of $\pi$-special characters of $G$ to the characters in $\text{Irr}_\pi(G)$ of $\pi$-degree.

We will be interested in the set of $\pi$-factorable characters of a $\pi$-separable group $G$. If $\alpha \in \text{Irr}(G)$ is $\pi$-special, and $\beta \in \text{Irr}(G)$ is $\pi'$-special, then we say the character $\alpha\beta$ is $\pi$-factorable, and we have the following important theorem:

**Theorem 4.2.** [13] Suppose that $G$ is a $\pi$-separable group, and $\alpha \in \text{Irr}(G)$ is $\pi$-special and $\beta \in \text{Irr}(G)$ is $\pi'$-special. Then $\alpha\beta$ is in $\text{Irr}(G)$, and the factorization is unique: if $\alpha_1\beta_1 = \alpha_2\beta_2$ (where $\alpha_1$ and $\alpha_2$ are $\pi$-special and $\beta_1$ and $\beta_2$ are $\pi'$-special), then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

It follows from the definition of $\pi$-factorable characters and the properties of $\pi$-special characters that if $N \triangleleft G$ and $\chi \in \text{Irr}(G)$ is $\pi$-factorable, then the constituents of $\chi_N$ are $\pi$-factorable.

4.2. Nuclei and vertices. We now use the notion of $\pi$-factorable characters to construct nuclei for ordinary irreducible characters in a manner very similar to
the normal nucleus of $\varphi \in \text{Irr}(G)$ discussed in Section 3. The subnormal nucleus discussed below and its properties were originally given by Isaacs in [16], while the normal nucleus was developed by Navarro in [30]. Since the constructions of the normal nucleus and the subnormal nucleus of a character $\chi \in \text{Irr}(G)$ are very similar, we will give both constructions at once.

Therefore, let $\chi \in \text{Irr}(G)$, where $G$ is $\pi$-separable, and we construct the normal (resp. subnormal) nucleus of $\chi$. If $\chi$ is $\pi$-factorable, define the normal (resp. subnormal) nucleus of $\chi$ to be the pair $(G, \chi)$. If $\chi$ is not $\pi$-factorable, then there exists a normal (resp. subnormal) subgroup $N$ (resp. $S$) of $G$ maximal with the property that some constituent $\gamma$ of $\chi_N$ (resp. $\chi_S$) is $\pi$-factorable. It can then be shown that the stabilizer of $\gamma$ is proper in $G$. (In the subnormal case, one must now show the very nontrivial result that a version of Clifford induction holds between the characters of the stabilizer of $\gamma$ in $N_G(S)$ lying over $\gamma$ and the characters of $G$ lying over $\gamma$.) Letting $\psi$ be the Clifford correspondent of $\chi$ in the stabilizer subgroup, we then recursively define the normal (resp. subnormal) nucleus of $\chi$ to be the normal (resp. subnormal) nucleus of $\psi$.

Notice two important properties of the nucleus pair $(U, \rho)$ constructed above. First, the nucleus character $\rho$ induces irreducibly to $\chi$. Secondly, notice that the construction process terminates when the Clifford correspondent constructed in some step is $\pi$-factorable. Thus the nucleus character $\rho$ is $\pi$-factorable.

Of course, choices were made in the above construction, and it is necessary to show that the nucleus pair $(U, \rho)$ is unique up to conjugacy. However, at each point in the construction process, the choices made were equivalent via conjugation (this is easy to see in the construction of the normal nucleus via Clifford theory, and is true but more difficult to prove in the case of the subnormal nucleus).

Before discussing the vertex pairs that arise from the above constructions, we briefly digress to discuss the connection between the above constructions and the Fong-Swan theorem. In [16], Isaacs defined the subset $B_\pi(G)$ of $\text{Irr}(G)$ by letting $B_\pi(G)$ consist of all of the irreducible characters of $G$ such that the subnormal nucleus character $\rho$ is $\pi$-special. Isaacs then shows that the restriction map $\chi \rightarrow \chi^\circ$ is a bijection from $B_\pi(G)$ onto $\text{Irr}(G)$. Similarly, Navarro [30] defined the set (which we will call $N_\pi(G)$) of $N_\pi$ characters by letting $N_\pi(G)$ consist of all of the irreducible characters of $G$ such that the normal nucleus character is $\pi$-special, and shows that the map $\eta \rightarrow \eta^\circ$ is a bijection from $N_\pi(G)$ to $\text{Irr}(G)$. (In [30] Navarro shows this only in the case that $\pi = p'$, though his proof immediately carries over to the more general $\pi$-case.) Interestingly, it seems as though one cannot prove Navarro’s result without using Isaacs’ result. Also, it is shown in [4] that $B_\pi(G) = N_\pi(G)$ if $G$ has odd order, but this equality is not true in general.

An ordinary irreducible character $\chi$ such that $\chi^\circ = \varphi \in \text{Irr}(G)$ is called a lift of $\varphi$, and we will have more to say about lifts in later sections, but for now we return to the construction of vertex pairs. Recall that one of the important properties of $\pi'$-special characters is that if $\beta$ is a $\pi'$-special character of $G$ and $Q$ is a Hall $\pi'$-subgroup of $G$, then $\beta_Q$ is in fact in $\text{Irr}(Q)$ and $\beta$ is the unique $\pi'$-special extension of $\beta_Q$ to $G$. Thus we have the following definition:

**Definition 4.3.** Let $G$ be a $\pi$-separable group and let $(U, \alpha \beta)$ be the subnormal nucleus of $\chi \in \text{Irr}(G)$, where $\alpha$ is $\pi$-special and $\beta$ is $\pi'$-special. If $Q$ is a Hall $\pi'$-subgroup of $U$, then we define the pair $(Q, \beta_Q)$ to be the subnormal vertex of $\chi$. 
Since the subnormal nucleus of $\chi$ is uniquely defined up to conjugacy in $G$, we have that the subnormal vertex of $\chi$ is uniquely determined up to conjugacy in $G$. One can similarly define the normal vertex of $\chi \in \text{Irr}(G)$, which we will call the Navarro vertex of $\chi$. Note then that we can paraphrase the definition of $B_\pi(G)$ as the set of irreducible characters of $G$ whose subnormal vertex character is trivial, and the definition of $N_\pi(G)$ can be similarly paraphrased. Since it is not true in general that $B_\pi(G) = N_\pi(G)$, then the subnormal vertex of $\chi \in \text{Irr}(G)$ need not be equal to the normal vertex.

There are however several other ways that one can “canonically” construct a pair $(U, \alpha\beta)$ from a character $\chi \in \text{Irr}(G)$, where $\alpha$ is $\pi$-special, $\beta$ is $\pi'$-special, and $(\alpha\beta)^G = \chi$. In addition, there could exist pairs with these properties that are not canonically constructed in any meaningful way.

**Definition 4.4.** Suppose that $G$ is $\pi$-separable and $\chi \in \text{Irr}(G)$. A nucleus for $\chi$ is a pair $(U, \alpha\beta)$, with $\alpha\beta \in \text{Irr}(G)$, such that $\alpha$ is $\pi$-special, $\beta$ is $\pi'$-special, and $(\alpha\beta)^G = \chi$. We say $(U, \alpha\beta)$ is a linear nucleus if, in addition, the $\pi'$-special factor $\beta$ is linear.

Thus a given character $\chi \in \text{Irr}(G)$ could have many nuclei, and the nuclei might not all be conjugate in $G$. We now define the vertex pair that corresponds to a nucleus.

**Definition 4.5.** Suppose $G$ is $\pi$-separable and $\chi \in \text{Irr}(G)$. Let $Q$ be a $\pi'$-subgroup of $G$ and suppose $\delta \in \text{Irr}(Q)$. We say that the pair $(Q, \delta)$ is a vertex pair for $\chi$ if there exists a nucleus $(U, \alpha\beta)$ for $\chi$ such that $Q$ is a Hall $\pi'$-subgroup of $U$ and $\beta_Q = \delta$, where $\beta$ is the $\pi'$-special factor of $\alpha\beta$. If $(Q, \delta)$ is a vertex pair for $\chi$ and $\delta$ is linear, we say that $(Q, \delta)$ is a linear vertex pair for $\chi$.

Note then that the definition of a vertex pair for an ordinary irreducible character $\chi$ of a $\pi$-separable group $G$ is very similar to the definition of a vertex subgroup of a character $\varphi \in \text{Irr}(G)$. The following theorem, which parallels some of the basic results about vertex subgroups discussed in Section 2, follows immediately from the above discussion:

**Theorem 4.6.** Suppose that $G$ is a $\pi$-separable group and let $\chi \in \text{Irr}(G)$. Then:

(a) There exists a vertex pair $(Q, \delta)$ for $\chi$.

(b) If $H \leq G$ and $\psi \in \text{Irr}(H)$ is such that $\psi^G = \chi$, then every vertex pair for $\psi$ is a vertex pair for $\chi$.

(c) If $(Q, \delta)$ is a linear vertex pair for $\chi$, then there is a Hall $\pi'$-subgroup $P$ of $G$ containing $Q$ such that $\chi(1)_{\pi'} = |P : Q|$.

(d) If $\chi^{\pi} = \varphi \in \text{Irr}(G)$, and if $(Q, \delta)$ is a linear vertex pair for $\chi$, then $Q$ is a vertex subgroup for $\varphi$.

Notice that linear vertex pairs for ordinary irreducible characters of $\pi$-separable groups have all of the properties that were discussed for vertex subgroups of irreducible $\pi$-partial characters except uniqueness. In fact, it is not true that if $(Q_1, \delta_1)$ and $(Q_2, \delta_2)$ are linear vertex pairs of $\chi \in \text{Irr}(G)$, then $(Q_1, \delta_1)$ is conjugate to $(Q_2, \delta_2)$. For instance, there exists a solvable group $G$ and a character $\chi \in B_\pi(G)$ (so that the subnormal vertex character is trivial) and such that $\chi \notin N_\pi(G)$ and the normal vertex for $\chi$ has a nontrivial linear character [4]. We will see in the next section, however, certain cases where the linear vertex pairs of $\chi \in \text{Irr}(G)$ are all conjugate.
4.3. **Inductive pairs.** Before moving on, however, we give another example, from [24], of how certain vertex pairs can be constructed, in this case from objects called inductive pairs. The purpose of this example is two-fold: first, we see yet another way to construct a nucleus pair, different from the normal or subnormal nucleus constructed above. Secondly, we will discuss in the next section uniqueness results related to this construction.

The study of inductive pairs was motivated by the following question about lifts of Brauer characters (or lifts of irreducible nuclei constructed above). Secondly, we will discuss in the next section uniqueness of these lifts with respect to normal subgroups. We say a set \( N = \{1 = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_k = G\} \) of normal subgroups of \( G \) is a \( \pi \)-chain if each factor \( N_i/N_{i-1} \) is either a \( \pi \)-group or a \( \pi' \)-group. We have the following definition from [24].

**Definition 4.7.** Suppose \( G \) is a \( \pi \)-separable group and \( N = \{1 = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_k = G\} \) is a \( \pi \)-chain for \( G \). Let \( V \subseteq G \) and \( \gamma \in \text{Irr}(V) \) be such that, for each \( i, \gamma \cap N_i \) is a multiple of some character \( \rho \in \text{Irr}(V \cap N_i) \), and \( (\rho N_i)^o \in \text{Irr}(N_i) \). Then we say the pair \((V, \gamma)\) is an **inductive pair** for \( N \).

Notice that if \((V, \gamma)\) is an inductive pair for \( N \), then \((\gamma^G)^o \in \text{Irr}(G)\), and the constituents of \( \chi_{N_i} \) are lifts of characters in \( \text{Irr}(N_i) \). In addition, it is an immediate consequence of Theorem 21.7 of [26] that \( \gamma \) must be \( \pi \)-factorable. By Theorem 3.2 of [24], it also follows that the \( \pi' \)-special factor of \( \gamma \) must in fact be linear. Thus the pair \((V, \gamma)\) must be a linear nucleus for \( \gamma^G \). Moreover, for each character \( \varphi \in \text{Irr}(G) \), it is known [25] that there necessarily exists an inductive pair \((V, \gamma)\) for \( N \) such that \((\gamma^G)^o = \varphi\).

Thus the inductive pairs give a “non-canonical” way of generating well-behaved lifts of \( \text{Irr}(G) \). We will see in Section 5.2 that the associated linear vertex pairs are unique (or at least, “almost” unique) up to conjugacy in \( G \).

### 5. Uniqueness results

Recall that the vertex subgroup of a character \( \varphi \in \text{Irr}(G) \) is uniquely determined up to conjugacy. However, given an ordinary irreducible character \( \chi \) of a \( \pi \)-separable group \( G \), it need not be the case that all of the vertex pairs associated to \( \chi \) are conjugate. In this section we examine conditions under which a uniqueness conclusion does hold.

#### 5.1. Uniqueness of vertex pairs of lifts in groups of odd order.

Recall that if \( \chi \in \text{Irr}(G) \) is such that \( \chi^o = \varphi \in \text{Irr}(G) \), then we say that \( \chi \) is a lift of \( \varphi \). Much of the study of lifts of Brauer characters can be characterized by seeking answers to the following question: What properties of Brauer characters are inherited by their lifts? In this section we will see the first of two results that show that if \( G \) has odd order, then the lifts of \( \varphi \in \text{Irr}(G) \) behave like \( \varphi \).

Our first important yet relatively easy result shows that if \( G \) has odd order, and \( \chi \in \text{Irr}(G) \) is a lift of \( \varphi \in \text{Irr}(G) \), then the vertex pairs of \( \chi \) all have linear vertex characters.
Theorem 5.1. [3] Suppose that $G$ has odd order and $\chi \in \text{Irr}(G)$ is such that $\chi^o = \varphi \in I_\pi(G)$. If $(Q, \delta)$ is a vertex pair for $\chi$, then $Q$ is a vertex for $\varphi$, and $\delta$ is linear.

Theorem 5.1, which is certainly not true if $G$ is not assumed to have odd order, is used to prove the main result of [7], which shows that if $G$ has odd order and $\chi \in \text{Irr}(G)$ is a lift of $\varphi \in I_\pi(G)$, then the set of vertex pairs for $\chi$ have the same uniqueness property as the vertex subgroups discussed in Sections 2 and 3.

Theorem 5.2. [7] Suppose that $G$ has odd order and $\chi \in \text{Irr}(G)$ is such that $\chi^o \in I_\pi(G)$. Then the vertex pairs for $\chi$ form a single conjugacy class.

Theorem 5.2 shows then that each vertex pair for a lift $\chi$ (in a group $G$ of odd order), regardless of whether or not it was constructed in some canonical manner, is uniquely defined up to conjugacy. This fact is especially useful in proving results about lifts using Clifford induction.

We now briefly discuss the proof of Theorem 5.2. The idea is to emulate, as much as possible, the proof that the vertex subgroups associated to irreducible $\pi$-partial characters are unique up to conjugacy. Only here, the role of “$\pi$-degree” for irreducible $\pi$-partial characters will be replaced by $\pi$-factorable for ordinary irreducible characters. Thus the goal is to show that if $\chi \in \text{Irr}(G)$ is such that $\chi^o \in I_\pi(G)$, and if $(Q, \delta)$ is any vertex pair for $\chi$, then $(Q, \delta)$ is conjugate to a vertex pair constructed from a normal nucleus for $\chi$. As in the proof of the uniqueness result discussed in Section 3, we begin by letting $U$ be a subgroup of $G$ and $\alpha \beta \in \text{Irr}(U)$ be $\pi$-factorable such that $(\alpha \beta)^G = \chi$, and $Q$ is a Hall $\pi'$-subgroup of $U$ and $\beta_Q = \delta$. Note that by Theorem 5.1, $\delta$ is linear (we point out that this is the first of two key parts of the proof where the odd order assumption is essential).

As in the proof of the uniqueness of vertex subgroups, we need to pick a specific normal subgroup $N$ and replace $U$ with $UN$. Here, we let $N$ be the unique normal subgroup of $G$ maximal with the property that the constituents of $\chi_N$ are factorable. It can easily be shown that $[UN:U]$ is a $\pi$-number. In order to replace the pair $(U, \alpha \beta)$ with the pair $(UN, (\alpha \beta)^{UN})$, we need to know that $(\alpha \beta)^{UN}$ is in fact $\pi$-factorable (note that this step in the proof regarding the uniqueness of vertex subgroups of irreducible $\pi$-partial characters, where we are using irreducible $\pi$-partial characters of $\pi$-degree instead of $\pi$-factorable characters, is trivial). It is indeed true that $(\alpha \beta)^{UN}$ is factorable and that if $\beta_1$ is the $\pi'$-special factor of $(\alpha \beta)^{UN}$, then $(\beta_1)_Q = \beta_Q$. However, this part of the proof again relies heavily on the fact that $G$ has odd order, and uses the results about the induction of $\pi$-special characters developed in [17]. The rest of the proof of Theorem 5.2 then follows by the same series of reductions as the proof of the uniqueness result for vertex subgroups of $I_\pi$-characters discussed in Section 3.

5.2. Uniqueness of vertex pairs associated to inductive pairs. We begin with the following definition:

Definition 5.3. Suppose that $G$ is a $\pi$-separable group, and that $\mathcal{N}$ is a collection of normal subgroups of $G$ with $G \in \mathcal{N}$. We say that $\chi \in \text{Irr}(G)$ is an $\mathcal{N}$-lift if, for each $N \in \mathcal{N}$, and each constituent $\psi$ of $\chi_N$, we have that $\psi^o \in I_\pi(N)$.

In Section 4.3, we discussed the notion of an inductive pair $(V, \gamma)$ for a $\pi$-chain $\mathcal{N}$ of normal subgroups of a $\pi$-separable group $G$, and we noted that if $\chi = \gamma^G$, then $\chi \in \text{Irr}(G)$ and $\chi$ is an $\mathcal{N}$-lift. Moreover, since $\gamma \in \text{Irr}(V)$ is $\pi$-factorable, we
know that we can define a vertex pair \((Q, \delta)\) associated to the inductive pair \((V, \gamma)\), and we know that \(\delta \in \text{Irr}(Q)\) is necessarily linear.

One might ask, then, if for a given \(\pi\)-chain \(\mathcal{N}\) and an \(\mathcal{N}\)-lift \(\chi\), are all of the vertex pairs associated to \(\chi\) conjugate? In other words, to what extent are the inductive vertex pairs associated to an \(\mathcal{N}\)-lift \(\chi\) unique? If we make the assumption that \(G\) is solvable and that \(\mathcal{N}\) is a chief series of \(G\), then the following result (from [9]) shows that the answer to the first question is “yes” in the case that \(2 \in \pi\) and “almost yes” in the case that \(2 \notin \pi\). (Recall that a sign character is a linear character \(\epsilon\) such that \(\epsilon^2 = 1\).)

**Theorem 5.4.** Suppose \(G\) is a solvable group and \(\mathcal{N}\) is a chief series of \(G\), and let \(\pi\) be a set of primes. Let \(\chi \in \text{Irr}(G)\) be an \(\mathcal{N}\)-lift. Then:

\(\begin{align*}
(\text{a}) & \quad \text{If } 2 \in \pi, \text{ then all of the inductive vertices of } \chi \text{ are conjugate.} \\
(\text{b}) & \quad \text{If } 2 \notin \pi, \text{ and if } (Q_1, \delta_1) \text{ and } (Q_2, \delta_2) \text{ are inductive vertices of } \chi, \text{ then there is an element } x \in G \text{ such that } Q_1^x = Q_2 \text{ and } (\delta_1)^x = \delta_2 \epsilon, \text{ where } \epsilon \text{ is a sign character of } Q_2.
\end{align*}\)

Actually, a slightly stronger statement is proven in [9]. We need only assume that \(G\) is \(\pi\)-separable and that each \(\pi\)-factor in the \(\pi\)-series \(\mathcal{N}\) is nilpotent. In fact, it is not yet known if this nilpotence assumption is truly necessary. Moreover, the character \(\epsilon\) in the statement of part (b) can be computed.

### 6. Lifts, Brauer characters, and groups of odd order

The Fong-Swan theorem states that if \(G\) is a \(p\)-solvable group and \(\varphi \in \text{IBr}_p(G)\), then there necessarily exists an ordinary irreducible character \(\chi \in \text{Irr}(G)\) such that \(\chi^o = \varphi\). This result of course extends to the characters in \(\text{I}(G)\), where \(G\) is a \(\pi\)-separable group. Until recently, however, not much was known about the set \(L_\varphi = \{\chi \in \text{Irr}(G) \mid \chi^o = \varphi\}\), other than \(L_\varphi\) is nonempty.

In this section we use some properties of vertex pairs to study the set \(L_\varphi\), where \(G\) is a \(\pi\)-separable group and \(\varphi \in \text{I}(G)\). First, we obtain lower bounds for the size of \(L_\varphi\), where \(\pi\) is arbitrary, and then we obtain upper bounds for the number of lifts of a Brauer character in a group of odd order.

#### 6.1. Lower bounds for the number of lifts

We begin by mentioning a result of Laradji [22] that gives a lower bound for the number of lifts of an irreducible 2-Brauer character of a solvable group in terms of “large” normal subgroups of \(G\). In particular, if \(Q\) is a vertex subgroup of \(\varphi \in \text{IBr}_2(G)\) and \(N\) is a normal subgroup of \(G\) such that \(QN/N\) is cyclic, then the number of lifts of \(\varphi\) is at least \(|QN:N|\).

We now sketch the argument for a lower bound for arbitrary \(\pi\)-separable groups. Suppose that \(\varphi \in \text{I}_\pi(G)\), where \(G\) is a \(\pi\)-separable group. We know from the results in Section 4.2 that there exists a unique ordinary irreducible character \(\chi \in \text{B}_\pi(G)\) such that \(\chi^o = \varphi\). We would like to use \(\chi\) to construct more lifts of \(\varphi\). To do this, we will use the linear characters of the vertex subgroup \(Q\) of \(\varphi\) that extend to the subnormal nucleus of \(\chi\). The key result is the following, from [29]:

**Theorem 6.1.** [29] Suppose \(G\) is \(\pi\)-separable, \(\chi \in \text{B}_\pi(G)\), and \((W, \alpha)\) is the subnormal nucleus of \(\chi\), so that \(\alpha\) is \(\pi\)-special. Then the map

\[
\beta \mapsto (\alpha \beta)^G
\]

is an injection from the \(\pi'\)-special characters of \(W\) into \(\text{Irr}(G)\).
In the above theorem, since \( \chi^o \in \text{Irr}(G) \) and \( \alpha^G = \chi \), then \( \alpha^o \in \text{Irr}(W) \). If \( \beta \in \text{Irr}(W) \) is linear and \( \pi' \)-special, then it is not hard to see that \( (\alpha\beta)^G \) is also a lift of \( \chi^o \), and thus the map \( \beta \to (\alpha\beta)^G \) is an injection from the set of linear \( \pi' \)-special characters of \( W \) into (but not onto — see \([3]\)) the lifts of \( \chi^o \). Thus we immediately have the following corollary:

**Corollary 6.2.** Let \( \varphi \in \text{Irr}(G) \), where \( G \) is \( \pi \)-separable, and suppose that \( \chi \in \text{Br}(G) \) is such that \( \chi^o = \varphi \). If \((W, \alpha)\) is a subnormal nucleus for \( \chi \), then

\[
|L_{\varphi}| \geq |W : W'||p'.
\]

Note that every Hall \( \pi' \)-subgroup \( Q \) of \( W \) is necessarily a vertex of \( \varphi \), and that the linear \( \pi' \)-special characters of \( W \) are in one-to-one correspondence with the linear characters of \( Q \) that extend to \( W \). Thus we see a connection between the “potential” vertex characters in the vertex subgroup of \( \varphi \) and the lifts of \( \varphi \). As noted before, however, the map constructed above is not surjective and thus does not yield all of the lifts of \( \varphi \). We also mention here that since these bounds were determined in terms of the nucleus subgroup, and Laradji’s bounds are in terms of the vertex subgroup, it is not known if there is any connection between the two. However, the two results are proved in a very different manner.

In the method described above, there is nothing particularly unique about \( \text{Br}(G) \) and the subnormal nucleus. In other words, one can obtain the same result by using the set \( \text{N}_{\pi}(G) \) and the normal nucleus \([3]\), or other similar constructions. For instance, in \([9]\) a similar lower bound for the number of \( N \)-lifts is developed, and in \([24]\) a similar result is obtained for generating lifts via objects called self-stabilizing pairs (defined in \([24]\)).

### 6.2. Upper bounds for the number of lifts in groups of odd order.

We are also able to use the vertex pairs to construct an upper bound for the number of lifts of a Brauer character in a group of odd order. We are switching from an arbitrary set of primes to \( \pi = \{p'\} \) because we require the nilpotence of the Hall \( \pi' \)-subgroups. It is likely (though not yet known) that much, if not all of the following arguments work if we allow \( \pi \) to be an arbitrary set of primes but require a Hall \( \pi' \)-subgroup to be nilpotent. The key to this argument is a certain map, defined by Navarro \([28]\), which we shall call the star map. For a group \( G \) of odd order and a \( p \)-subgroup \( Q \) and a character \( \delta \in \text{Irr}(Q) \), we define the set \( \text{Irr}(G|Q, \delta) \) to be the set of characters \( \chi \in \text{Irr}(G) \) that have Navarro vertex \((Q, \delta)\) (Recall that the Navarro vertex is the vertex obtained from the normal nucleus construction). In addition, we let \( G_{\delta} \) be the stabilizer of \( \delta \) in \( N_{G}(Q) \), and we say a character \( \mu \in \text{Irr}(G_{\delta}|\delta) \) has relative defect zero, denoted by \( \mu \in rdz(G_{\delta}|\delta) \), if

\[
\frac{\mu(1)_p}{\delta(1)_p} = |G_{\delta} : Q|_p.
\]

Navarro’s key result about the star map is the following:

**Theorem 6.3.** \([28]\) Let \( G \) be a group of odd order, and let \( Q \) be a \( p \)-subgroup of \( G \) and \( \delta \in \text{Irr}(Q) \). Then there exists a well-defined injection \( \chi \mapsto \chi^* \) from the set \( \text{Irr}(G|Q, \delta) \) into \( rdz(G_{\delta}|\delta) \).

To apply Navarro’s star map to the study of lifts, we need to understand the behavior of lifts with respect to the star map \([3]\). There are two key observations here. First, suppose that \( \chi \in \text{Irr}(G|Q, \delta) \) is a lift of \( \varphi \in \text{IBr}_{p}(G) \). Then \( \chi^* \in
\[ rdz(G|\delta) \] is a lift, and in fact \((\chi^*)_{N_G(Q)}\) is a lift of a uniquely defined character \(\tilde{\varphi} \in IBr_p(N_G(Q)|Q)\). Secondly, if \(\chi, \psi \in \text{Irr}(G|Q, \delta)\) are such that \((\chi^*)^o = (\psi^*)^o \in IBr_p(G_\delta)\), then \(\chi = \psi\).

Combining the above two results, it is not too difficult to prove:

**Lemma 6.4.** Suppose \(G\) be a group of odd order and \(\varphi \in IBr_p(G)\) has vertex subgroup \(Q\), and let \(\delta\) be a linear character of \(Q\). Then the map \(\chi \mapsto (\chi^*)_{N_G(Q)}\)

is an injection from the set of lifts of \(\varphi\) in \(\text{Irr}(G|Q, \delta)\) to the set of lifts of \(\tilde{\varphi}\) in \(\text{Irr}(N_G(Q)|Q, \delta)\). Moreover, the number of lifts of \(\varphi\) in \(\text{Irr}(G|Q, \delta)\) is bounded above by \(|N_G(Q) : G_\delta|\).

Because \(G\) has odd order, we know from Section 5 that the vertex character for every lift in a group of odd order must be linear. The odd order assumption is also needed to define the star map above. It is not yet known if the above lemma, or some variation thereof, is true for arbitrary solvable groups.

An easy consequence of the above lemma (by simply summing over the \(N_G(Q)\) classes of linear characters of \(Q\)) is the following upper bound on the number of lifts of a given Brauer character in a group \(G\) of odd order:

**Theorem 6.5.** [3] Suppose that \(G\) is a group of odd order, and \(\varphi \in IBr_p(G)\) has vertex subgroup \(Q\). Then \(|L_\varphi| \leq |Q : Q'|\).

**7. Vertices and normal subgroups**

In this section we will examine the relationship between vertex subgroups and vertex pairs and normal subgroups. We will then see how these results can be used to develop further properties of lifts of Brauer characters.

**7.1. Restriction of irreducible \(\pi\)-partial characters to normal subgroups.** The following result of Laradji [23] shows that when restricting to normal subgroups, the vertex subgroups of irreducible \(\pi\)-partial characters behave exactly as one would hope.

**Theorem 7.1.** [23] Let \(G\) be a \(\pi\)-separable group with normal subgroup \(N\), and suppose that \(\varphi \in I_\pi(G)\) has vertex \(Q\). Then some constituent of \(\varphi_N\) has vertex subgroup \(Q \cap N\).

Theorem 7.1 can be used to prove a strengthening for the Alperin weight conjecture in groups of odd order. Recall from Section 3.3 that Isaacs and Navarro [21] proved a version of the Alperin weight conjecture for solvable groups “one vertex at a time”. Specifically, they showed that if a Hall \(\pi\)'-subgroup for the \(\pi\)-separable group \(G\) is nilpotent, then for each \(\pi\)'-subgroup \(Q\) of \(G\), we have

\[ |I_\pi(G|Q)| = |w(Q)|, \]

where \(I_\pi(G|Q)\) is the set of characters in \(I_\pi(G)\) with vertex subgroup \(Q\) and \(w(Q)\) is the set of weights of \(G\) associated to the \(\pi\)'-subgroup \(Q\). However, no explicit bijection was given in the case that \(G\) is \(\pi\)-separable.

If, however, \(G\) has odd order and \(\pi' = p\), Navarro has shown [31] that one can construct an explicit bijection from \(IBr_p(G|Q)\) to \(w(Q)\), which we will denote by \(\varphi \rightarrow \tilde{\varphi}\) (this is the same \(\tilde{\varphi}\) mentioned in Section 6.2). To strengthen the Alperin weight conjecture in this case to take into account normal subgroups, note that if
Theorem 7.2. [8] Suppose that $G$ is a group of odd order with normal subgroup $N$, and let $Q$ be a $p$-subgroup of $G$ and $P = Q \cap N$. If $\varphi \in \text{IBr}_p(G|Q)$ and $\theta \in \text{IBr}_p(N|P)$, then $\varphi$ lies over $\theta$ if and only if $\varphi_1 \in \text{IBr}_p(N_G(P)|Q)$ lies over $\tilde{\theta} \in \text{IBr}_p(N_N(P)|P)$. Thus for a fixed character $\theta \in \text{IBr}_p(N|P)$, the map

$$\varphi \mapsto \varphi_1$$

is a bijection from

$$\{ \varphi \in \text{IBr}_p(G|Q) \mid [\varphi, \theta] \neq 0 \}$$

to

$$\{ \psi \in \text{IBr}_p(N_G(P)|Q) \mid [\psi_{N_P}(P), \tilde{\theta}] \neq 0 \}.$$  

It turns out that actually more is true. In [5], it is shown that with the above notation, $[\varphi, \theta] = [\psi_{N_P}(P), \tilde{\theta}]$. Of course, if $N = G$, then $P = Q$ and $N_G(P) = N_G(Q)$, and thus $\varphi_1 = \tilde{\varphi}$, which is the specific bijection for the Alperin weight conjecture in groups of odd order.

7.2. Restriction of lifts to normal subgroups. In order to examine the behavior of lifts of Brauer characters with respect to normal subgroups in groups of odd order, we need to first examine the behavior of the Navarro vertices with respect to normal subgroups. The following result, from [6], is proven in a manner very similar to that of Theorem 7.1.

Theorem 7.3. [6] Let $G$ be a group of odd order, and let $\chi \in \text{Irr}(G)$ be a lift of $\varphi \in \text{IBr}_p(G)$, and suppose that $\chi$ has Navarro vertex $(Q, \delta)$. Suppose $N \lhd G$. Then there is a constituent $\psi$ of $\chi_N$ such that $\psi$ has normal vertex $(Q \cap N, \delta_{Q\cap N})$.

Note that the above theorem does not say that the constituents of $\chi_N$ have to be lifts of Brauer characters. In fact, the constituents of $\chi_N$ need not be lifts of Brauer characters. The above theorem is also not true if one removes either the assumption that $G$ has odd order or the assumption that $\chi$ is a lift. However, the following is an easy corollary of the above result:

Corollary 7.4. [6] Let $G$ be a group of odd order, and suppose that $\chi \in \text{Irr}(G)$ is a lift of a Brauer character. If $N \lhd G$ is such that $G/N$ is a $p$-group, then the constituents of $\chi_N$ are lifts. Moreover, if $\psi \in \text{Irr}(N)$ is a constituent of $\chi_N$ such that $\psi^\circ = \theta$, then $G_\psi = G_\theta$.

The above result leads naturally to the question of determining for which normal subgroups $N$ of a group $G$ of odd order are the constituents of $\chi_N$, for a lift $\chi \in \text{Irr}(G)$, also lifts? It is easy to create examples where these constituents are not lifts. Ideally, one would like to have an answer in terms of the “local” information related to the lift $\chi$. The following result gives a sufficient condition, in terms of the local behavior of the Navarro vertex, for the constituents of a lift $\chi$ restricted to a normal subgroup $N$ to also be lifts. Note that the sufficient condition depends only on the local behavior of the Navarro vertex.
Theorem 7.5. [5] Suppose that $G$ is a group of odd order, and $\chi \in \text{Irr}(G)$ is a lift with Navarro vertex $(Q, \delta)$. Suppose $N \lhd G$ and set $(P, \lambda) = (Q \cap N, \delta_{Q \cap N})$. If $\lambda$ is invariant in $N_G(P)$, then the constituents of $\chi_N$ are lifts.

8. Cyclic vertex subgroups and some open questions

8.1. Brauer characters of solvable groups with cyclic vertex subgroups. There is a well developed yet complicated theory about blocks with cyclic defect groups (see [10] or [12], for instance). In particular, much is known about the decomposition numbers and generalized decomposition numbers in the block. It is easy to show that if $D$ is a defect group of a block $B$ of a solvable group $G$, and if $\varphi \in \text{IBr}_p(B)$, then some vertex subgroup of $\varphi$ is contained in $D$. It is perhaps surprising, though, that if $\varphi$ has a cyclic vertex subgroup, then $Q = D$ and therefore we may apply the results about blocks with cyclic defect group to determine the lifts of $\varphi$ (see [2] for a proof that $Q = D$ if $G$ is solvable). In fact, the conclusion that $Q = D$ if $Q$ is cyclic holds even without the assumption that $G$ is solvable (see [11]). The following result is immediate using the standard results about blocks with cyclic defect group and the Fong-Swan theorem:

Theorem 8.1. Suppose $G$ is a $p$-solvable group and $(|G|, p - 1) = 1$. If $\varphi \in \text{IBr}_p(G)$ is in the block $B$ and has cyclic vertex subgroup $Q$ of order $p^d$, then $\varphi$ is the unique Brauer character in $B$, and $|L_\varphi| = |\text{Irr}(B)| = p^d$.

Using some of the results from Section 7, one can obtain similar results for some special cases without using the theory of cyclic defect groups. In particular, if $G$ has odd order and $\varphi$ is a Brauer character of $G$ with a vertex subgroup of order $p$, with $(|G|, p - 1) = 1$, then it is shown in [5] that $|L_\varphi| = p$. Moreover, in this case, if $N$ is any normal subgroup of $G$, and $\chi$ is a lift of $\varphi$, then every constituent $\psi$ of $\chi_N$ is a lift.

8.2. Open questions. The above results leave open many questions regarding the behavior of vertex subgroups and vertex pairs in solvable groups.

For instance, many of the results above are true for groups of odd order, but it is not yet known if they are true for arbitrary solvable (or $\pi$-separable) groups. Can one determine an upper bound for the number of lifts of a Brauer character or irreducible $\pi$-partial character like the bounds obtained in Section 6? Is there any way to extend the character correspondence for the Alperin weight conjecture in groups of odd order to arbitrary solvable groups?

There are also many open questions regarding characters with cyclic vertex subgroups. Does the theory developed in the previous subsection generalize to irreducible $\pi$-partial characters of $\pi$-separable groups? And can one obtain a complete characterization of lifts in this case that is independent of the theory of cyclic defect groups?

Even if we restrict our attention to groups of odd order, there remain many open questions. We have upper and lower bounds for the number of lifts of a Brauer character in groups of odd order, but we do not yet have an exact count. Is an exact count in terms of the “local” behavior of the Navarro vertices possible? Also, some of the results of [28] indicate that there may be a stronger relation between the structure of the set of lifts of Brauer characters in a group of odd order and the local structure. What exactly is this relationship? The results of [5] also indicate that the behavior of lifts of Brauer characters with respect to normal subgroups...
of $G$ may be determined by local properties of the Navarro vertices, though the precise nature of that relationship remains unknown.

References


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