Here are some practice questions for the second exam. Some of them are lifted from the book, so they aren’t “original”. Then again, when I write the exam, I usually have the book in front of me. So make of that what you will. Also, this practice exam is much longer than what you will see Friday.

(1) Of the groups listed below, three of them are isomorphic to each other, and none of the rest are isomorphic to any of the others. Circle the three that are isomorphic to each other.

   (a) $U(5)$
   (b) $\text{Aut}(\mathbb{Z}_5)$
   (c) $\text{Aut}(\mathbb{Z}_4)$
   (d) $\mathbb{Z}_5$
   (e) $\text{Aut}(\mathbb{Z}_{10})$
   (f) $\text{Aut}(\mathbb{Z}_8)$

(2) Some (possibly more than one) of the properties listed below are NOT included in the definition of a ring $R$. Circle the statements below that are NOT part of the definition of a ring.

   (a) $R$ is closed under multiplication.
   (b) $R$ has a multiplicative identity.
   (c) $a + b = b + a$ for all $a, b \in R$.
   (d) $a \cdot b = b \cdot a$ for all $a, b \in R$.
   (e) For every $r \in R$, there exists an element $s$ such that $r \cdot s = 1$. 
(3) Suppose $R$ is a ring and let $x$ be an element of $R$. Define the subset $S$ of $R$ by

$$S = \{ a \in R | ax = 0 \}.$$

Prove that $S$ is a subring of $R$.

(4) Suppose that $G$ and $H$ are groups and that $\varphi : G \to H$ is an isomorphism. Let $g \in G$ and let $h = \varphi(g)$. Prove that

$$\varphi(C_G(g)) = C_H(h).$$

(5) Suppose that $H$ is a subgroup of $S_n$ of odd order. Prove that $H$ is contained in $A_n$.

(6) Prove or disprove that $U(20)$ and $U(24)$ are isomorphic.

(7) Suppose that $G$ and $H$ are groups, and that $\varphi : G \to H$ is an isomorphism. Show that if $z \in Z(G)$, then $\varphi(z) \in Z(H)$.

(8) Suppose that $G$ is a finite group of order $n$ and that $m$ is relatively prime to $n$. If $g \in G$ is such that $g^m = 1$, show that $g = 1$. 

(9) Show that if \( R \) is a ring such that the additive group \((R, +)\) is cyclic, then \( R \) is a commutative ring.

(10) Give an example of a group \( G \) with a subgroup \( H \) such that \(|G : H| = 3\), but the left cosets are \textbf{not} the same as the right cosets. (Hint: there is a group of order 6 that works).