Math 415/515
Review for Final Exam

Essentially I wrote two tests worth of questions. Here are the ones that didn’t make it onto the test, in no particular order. Some of these are maybe a little longer than I would probably put on a test, but if you can do these, you should be able to do the ones on the exam. Also, some of them are perhaps a little shorter. Enjoy.

(1) How many ways are there to arrange the letters in the word SESSIONS?

(2) How many ways are there to form a sequence of ten letters from 4 A’s, 4 B’s, 4 C’s, and 4 D’s if each letter must appear at least twice? (Hint: you might want to break this problem into cases.)

(3) How many ways are there to assign 100 different diplomats to five different countries? How many ways if exactly 20 diplomats must be assigned to each country?

(4) How many arrangements of the letters a, e, i, o, u, x, x, x, x, x, x, x, x are there (8 x’s) if no two vowels can be consecutive?

(5) Give an argument to show that

\[ \sum_{k=0}^{m} \binom{m}{k} \binom{n}{r+k} = \binom{m+n}{m+r}. \]
(6) Find the generating function for the number of solutions to

\[ e_1 + e_2 + e_3 + e_4 = r \]

where each \( e_i > 0 \) and \( e_2 \) and \( e_4 \) are odd.

(7) How many ways are there to get a sum of 30 when 12 distinct (i.e. differently colored) dice are rolled?

(8) Use an exponential generating function to find the number of ways to place 25 people into three rooms with at least one person in each room.

(9) Solve the recurrence relation

\[ a_n = 2a_{n-1} + 3a_{n-2} \]

with the initial conditions \( a_0 = 1 \) and \( a_1 = 1 \).

(10) Use generating functions to solve the recurrence relation

\[ a_n = a_{n-1} + n \]

with the initial condition \( a_0 = 1 \).
(11) Show that $p_n$, the number of ways to legally place $n$ pairs of parentheses, satisfies the equation

$$p_{n+1} = \sum_{k=0}^{n} p_k p_{n-k}.$$ 

Use this to show that the generating function for $p_n$ is given by $h(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$.

(12) Use inclusion-exclusion to compute the number of integer solutions to

$$x_1 + x_2 + \ldots + x_6 = 20$$

with $0 \leq x_i \leq 8$ for each $i$.

(13) Given a connected graph $G$ with $n$ vertices, $e$ edges, and $r$ regions, show that the sum of the degrees of the vertices is $2e$ and the sum of the number of edge curves that bound each region is $2e$. Also, show that the number of regions bounded by an odd number of edge curves must be odd.

(14) State and prove Euler’s formula.

(15) Know the results on closed Eulerian paths and the results on Hamiltonian cycles and be prepared to use them.
(16) Let $G$ be the graph given as follows: $G$ has 64 vertices, each corresponding to a square on a chess board. Two vertices are connected by an edge if and only if the two squares on the chessboard are adjacent. Is $G$ bipartite? (Hint: think about the colors of the squares). Use this to answer the following question: Is it possible for a knight to travel (using legal knight moves only) from one corner of the chessboard to the opposite corner (for instance from the bottom left corner to the top right corner) while visiting every square on the chessboard exactly once?

(17) Know the equivalent conditions on a graph $G$ in order for $G$ to be a tree.

(18) Use the ideas in one of the “planar graph” proofs we did in class to show that the complete graph $K_5$ is not planar. Use this to show that $K_n$ is not planar for $n \geq 5$. 