Math 611
Possible Project Topics

Here is a list of possible topics for the projects at the end of the semester. There are of course lots of very good topics not on this list, you’re welcome to do anything you feel like, as long as (1) It’s about or related to groups and their applications, and (2) I approve it. In fact, talk to me first regardless of whether you plan on doing something from this list or something else.

1. **Chermak-Delgado subgroups.** These subgroups, which have a relatively simple definition, are useful tools for creating certain normal subgroups, and thus proving certain groups are not simple. There is an interesting discussion in Chapter 1, Section G of Isaacs book, and a paper of Glauberman that expands on it.

2. **Maximal subgroups of the symmetric groups** For the symmetric group $S_n$, there are some subgroups that are obviously maximal subgroups, and some weird not-so-obvious maximal subgroups. There is a relatively easy paper by a couple of old grad school friends of mine (Bret Benesh and Ben Newton) that characterizes them.

3. **The automorphism tower theorem** Start with a group $G$, and let $G_1$ be the automorphism group of $G$. Let $G_2$ be the automorphism group of $G_1$, and recursively define $G_n$ like this. There is a nice result that says that if $G$ has a trivial center, then this must eventually terminate.

4. **Minimal simple groups** Read Kendall’s thesis. (Sorry, Kendall, you don’t get to do this project.) This discusses some properties of groups that are “barely” simple, and is related to Burnside’s theorem, which says that any group of order $p^aq^b$ is solvable.

5. **Inner automorphisms of the symmetric groups** It is not too difficult to show that if $n \neq 6$, then every automorphism of $S_n$ is “inner”, in other words every automorphism of $S_n$ is conjugation. It is interesting that $S_6$ has an automorphism that doesn’t come from conjugation, but no other symmetric group does.

6. **Applications to physics and/or chemistry** There are many, many applications of group theory to physics and chemistry. I, however, know none of them. So enlighten me. (I’m pretty sure things like the Heisenberg group, unitary groups, and spin groups come up. Show me what they are and how they apply).

7. **Braid groups and/or braid group cryptography** Braid groups are interesting infinite analogues of the symmetric groups. They also are “complex” enough to admit a Diffe-Hellman protocol, which is a cryptographic method that makes your credit card secure.
(8) **Hall \( \pi \)**-subgroups If we replace the prime \( p \) with a set of primes \( \pi \), then a Sylow \( p \)-subgroup becomes a Hall \( \pi \)-subgroup. However, for an arbitrary finite group \( G \), these are not guaranteed to exist (unlike Sylow subgroups). The good news is, if the group \( G \) is solvable, they are guaranteed to exist, and the natural analogues of the Sylow C, D, and E theorems hold.

(9) **Longest elements of the symmetric group** We saw that any element of \( S_n \) can be written as a product of transpositions. It is not hard to take this one step further and see that any element of \( S_n \) can be written as a product of transpositions of the form \((i, i + 1)\). Some elements of \( S_n \) are “longer” than others when doing this, and we can classify the “longest” elements.

(10) **What are the Mathieu groups and we do we care?** The five Mathieu groups are among the simplest of the sporadic simple groups, and were the first sporadic simple groups discovered. They were in some sense the jumping off point for the classification of finite simple groups, and they are also related to interesting sphere-packing problems.

These are just 10 random topics I came up with off the top of my head. As I said before, anything related to finite (or even infinite) groups is a good topic, as long as you run it by me first. If you want to know more about these topics, or want to explore other topics, come talk to me.