Irreducible representations of the symmetric group

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- I'm interested in representations of finite groups...
- ...but combinatorics is fun and is accessible to everyone.
- Somewhere between the two lies the representation theory of the symmetric group.

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$$\{1, 2, \ldots, n\}$$

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We will use the standard cycle notation: the function π defined by $\pi(1) = 5, \pi(5) = 2, \pi(2) = 7, \pi(7) = 1, \pi(4) = 6$, and $\pi(6) = 4$ is denoted by the element

$$\pi = (1527)(46) \in S_7.$$

Note that 3 is fixed under this permutation, and we omit 3 when we write π.

We compose functions from left to right, so that if $\pi = (1527)(46)$ and $\rho = (134)(256)$, then

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The conjugacy classes of S_n will be of great interest to us, so we point out that in S_n, two elements π and ρ are conjugate if and only if they have the same cycle type - that is, if for each positive integer k, there are the same number of cycles of length k in π and of length k in ρ.

Thus, for instance, the elements (1345)(678) and (1923)(457) are conjugate in S_9.

If we insist on ordering our cycles from largest to smallest, then we can associate to each conjugacy class of S_n a **partition** λ of n, which is a nonincreasing sequence $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ of nonnegative integers such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

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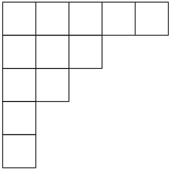
Thus (1345)(678) corresponds to the partition $\{4, 3, 1, 1\}$ of 9 (note that 2 and 9 are fixed).

We can associate a **Young diagram** to each partition of n - for example, the partition $\lambda = (5, 3, 2, 1, 1)$ of 12 has the associated Young diagram $T_\lambda =$



Thus we have the following natural correspondence:

$$\{\text{Conjugacy classes of } S_n\} \longleftrightarrow \{\text{Young diagrams of size } n\}$$

We are interested in the representations of the symmetric group. For now, G is any finite group, and $\mathbb{F}$ is any field. All of our vector spaces will be assumed to be finite dimensional.

Definition

A **representation** $\mathcal{X}$ of G of degree n over $\mathbb{F}$ is a homomorphism

$$\mathcal{X} : G \to GL_n(\mathbb{F}).$$

In this case, we say that n is the degree of the representation.

Note, then, that any representation of G over a field $\mathbb{F}$ naturally induces an action of G on $\mathbb{F}^n$. We then have the following useful equivalent definition:

Definition

If G is a group and V is a vector space, then we say that V is a G-module, or a representation of G, if G acts on V. (Recall that this just means that $1 \in G$ acts trivially, and for $v \in V$ and $g, h \in G$, we have $(v \cdot g) \cdot h = v \cdot (gh)$).

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- Representations tell us interesting things about groups (Odd-order theorem, classification of simple groups).
- Representations tell us interesting things about lots of other topics (wireless network design, probabilities of card-shuffling).

Sometimes a representation is too much information to keep track of, but we can still get most of the useful information by looking at the associated character:

Definition

If $\mathcal{X}$ is a representation of degree n of the finite group G over the field $\mathbb{F}$, then the **character** χ associated with $\mathcal{X}$ is just the composition of $\mathcal{X}$ with the trace map, i.e. $\chi(g) = tr(\mathcal{X}(g))$.

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Since the trace is invariant under conjugation by invertible matrices, then characters are constant on conjugacy classes.

Some representations appear to be different, but are actually the same:

Definition

Two representations $\mathcal{X}$ and $\mathcal{Y}$ are **equivalent** if there exists an invertible matrix M such that, for all $g \in G$, we have

$$M^{-1}\mathcal{X}(g)M = \mathcal{Y}(g).$$

Equivalently, we say two G-modules V and W are equivalent if there exists an invertible linear map $T : V \to W$ such that $T(v \cdot g) = T(v) \cdot g$ for all $g \in G$.

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Notice that equivalent representations afford the same character.

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Notice that if χ is a character afforded by an irreducible representation, then χ cannot be written as $\alpha + \beta$ for characters α and β.

Let's look at an example: Let $G = S_n$, and let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$.
Let S_n act on $\mathbb{C}^n$ in the natural way: if

$$v = \alpha_1 e_1 + \cdots + \alpha_n e_n,$$

then

$$v \cdot \pi = \alpha_1 e_{\pi(1)} + \cdots + \alpha_n e_{\pi(n)}.$$

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For instance, if $n = 4$, then under this representation,

$$(143) \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This does not give us an irreducible representation. Notice that the subspace W defined by $W = \{v \in \mathbb{C}^n | \alpha_1 = \alpha_2 = \ldots = \alpha_n\}$ is fixed by S_n.

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However, W is irreducible (it has dimension 1), and the complement,

$$W^{\perp} = \left\{ v \in V \middle| \sum_{k=1}^{n} \alpha_k = 0 \right\}$$

is irreducible of degree $n - 1$.

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- If χ is an irreducible character, then the degree of χ divides the order of the group.
- There are exactly as many irreducible characters of G as there are conjugacy classes of G.
- In fact, the irreducible characters of G form a basis for the vector space of class functions on G.

Unfortunately, despite the fact that the set of conjugacy classes and the set of irreducible characters of G (over $\mathbb{C}$) have the same size, there is no nice way to associate one with the other....

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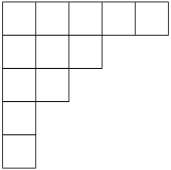
If $\mathbb{F}$ is any field, and λ is any partition of n, then there is a "natural" way to associate to λ a unique representation S^λ of S_n.

We will very briefly outline how that works. If λ is a partition of n, then we have seen that there is a Young diagram T_λ associated to λ. For instance, if $\lambda = \{5, 3, 2, 1, 1\}$, then T_λ is



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2	7	8	3	11
6	1	9		
12	5			
4				
10				

Notice that S_n acts naturally on the set of tableau of shape λ.

A tableau T is **standard** if the numbers in the boxes increase across rows and down columns.

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1	3	4	5	6
2	7	12		
8	10			
9				
11				

is a standard tableau.

We can associate to the partition $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ the vector space $\mathbb{F}^m$, where

$$m = \frac{n!}{\lambda_1!\lambda_2!\ldots\lambda_k!}.$$

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- If $\lambda = \{n-1, 1\}$, and $\mathbb{F} = \mathbb{C}$, then S^λ is $W^\perp$, defined as before.
- If $\lambda = \{5, 3, 2, 1, 1\}$, and $\mathbb{F} = \mathbb{C}$, then S^λ is a representation of degree 7700.

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Since there are as many conjugacy classes as there are partitions, and there are as many irreducible representations (over $\mathbb{C}$) as there are conjugacy classes, then the Specht modules over $\mathbb{C}$ are exactly all of the irreducible representations over $\mathbb{C}$.

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Thus we have, if $\mathbb{F} = \mathbb{C}$

$$\{\text{conj classes of } S_n\} \leftrightarrow \{\text{partitions } \lambda \text{ of } n\} \leftrightarrow \{\text{Specht modules } S^\lambda\}$$

where each arrow is "natural".

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Thus, given a Specht module S^λ, we can get a characteristic p representation (i.e. a representation over $\mathbb{F}_p$) by simply reducing the entries of the matrices modulo p.

The mod-p reduction of S^λ, however, will almost always **not** be irreducible as a representation of S_n over $\mathbb{F}_p$.

For instance, if $\lambda = \{5, 3, 2, 1, 1\}$, then when reducing the irreducible Specht module S^{λ} modulo 3, then this representation breaks down as 18 smaller representations.

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However, every now and then, an irreducible representation of S_n over $\mathbb{C}$ does stay irreducible when reduced modulo p. For instance, if $\mu = \{5, 2, 1, 1, 1, 1, 1\}$, then S^μ is irreducible when the entries are reduced mod 3.

This, then, is our motivating question:

Given a positive integer n and a prime p, then what is a necessary and sufficient condition on the partition λ to guarantee that the Specht module S^{λ} (over $\mathbb{C}$) remains irreducible when reduced modulo p to a representation over $\mathbb{F}_p$?

A solution to this problem was conjectured by James and Mathas in the late 1970's, and was proven by Fayers in 2005. To understand the solution, we need to look at the **hook lengths** of a partition.

Definition

If (a, b) denotes the box in row a and column b of the Young diagram of the partition λ, then the hook length $h_\lambda(a, b)$ is defined to be the number of boxes directly to the right and directly below (a, b) (including the box (a, b) itself).

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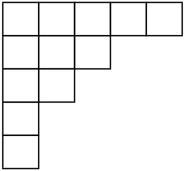

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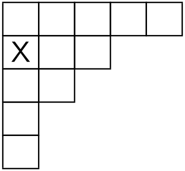
Let $\lambda = \{5, 3, 2, 1, 1\}$.
The box $(2, 1)$:



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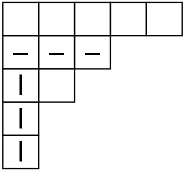
The hook length of box $(2, 1)$...



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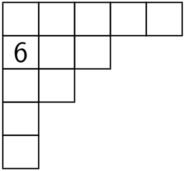
Let $\lambda = \{5, 3, 2, 1, 1\}$.
The hook length of box $(2, 1)$... is 6.



Or better yet, a definition by example:

Let $\lambda = \{5, 3, 2, 1, 1\}$.
Now we put in all of the hook lengths.

9	6	4	2	1
6	3	1		
4	1			
2				
1				

As an aside, notice that λ is a partition of 12, and the product of the hook lengths is

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which is the dimension of the corresponding Specht module.

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So one might conjecture (correctly), that for any partition λ, the dimension of the corresponding Specht module S^λ (over $\mathbb{C}$) is

$$\dim(S^\lambda) = \frac{n!}{\text{product of all of the hook lengths in } \lambda}.$$

Getting back to our main question: What conditions on the partition λ guarantee that the Specht module S^λ (over $\mathbb{C}$) remains irreducible when reduced modulo p?

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For now, we'll let $p = 3$ and we'll look at partitions of 52:

- If $\lambda = \{16, 14, 9, 7, 2, 2, 1, 1\}$, then S^λ remains irreducible when reduced modulo 3.

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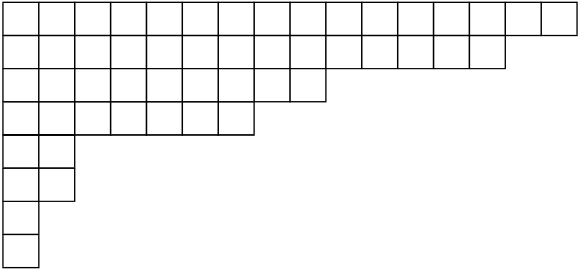
For now, we'll let $p = 3$ and we'll look at partitions of 52:

- If $\lambda = \{16, 14, 9, 7, 2, 2, 1, 1\}$, then S^λ remains irreducible when reduced modulo 3.
- But if $\mu = \{16, 14, 9, 7, 3, 2, 1\}$, then S^μ does not remain irreducible when reduced modulo 3.

The tableau for λ:



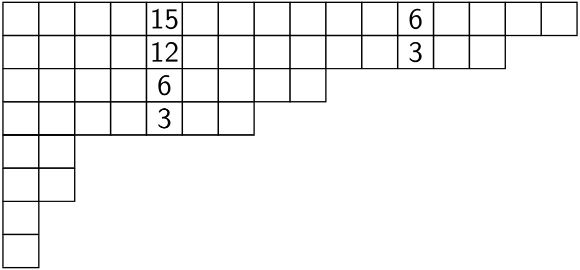
Recall S^λ is irreducible when reduced modulo 3.

We put in the hook lengths for T_λ:

23	20	17	16	15	14	13	11	10	8	7	6	5	4	2	1
20	17	14	13	12	11	10	8	7	5	4	3	2	1		
14	11	8	7	6	5	4	2	1							
11	8	5	4	3	2	1									
5	2														
4	1														
2															
1															

Recall S^λ is irreducible when reduced modulo 3.

Too much information. Take out everything not divisible by 3.



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Let's look at S^μ:

22	20	18	16	15	14	13	11	10	8	7	6	5	4	2	1
19	17	15	13	12	11	10	8	7	5	4	3	2	1		
13	11	9	7	6	5	4	2	1							
10	8	6	4	3	2	1									
5	3	1													
3	1														
1															

Recall that S^μ is **not** irreducible when reduced mod 3.

We'll ignore everything not divisible by 3:

		18		15							6				
		15		12							3				
		9		6											
		6		3											
	3														
3															

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Now, before we can state our main result, we need to make a couple of definitions:

Definition

Let n be a positive integer, and p a prime. If $n = p^a b$, where $(p, b) = 1$, then we define $v_p(n) = a$.

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Definition

If λ is a partition of n, with the corresponding Young diagram T_λ, and if (a, b) is a box in T_λ, we say that (a, b) is **bad** if there is a box (a, y) in the same row as (a, b) and a box (x, b) in the same column as (a, b) such that $v_p(h_\lambda(a, y)) \neq v_p(h_\lambda(a, b))$ and $v_p(h_\lambda(x, b)) \neq v_p(h_\lambda(a, b))$.

Note that T_λ did not contain any bad boxes, and T_μ did contain bad boxes.

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Theorem

(Fayers, 2005) Suppose λ is a partition of n and p is an odd prime. If S^λ is the corresponding irreducible Specht module over $\mathbb{C}$, then S^λ remains irreducible when reduced modulo p if and only if T_λ contains no bad boxes.

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Theorem

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The theorem is mostly true if $p = 2$.

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General question from representation theory: If χ is an ordinary irreducible character of a group G (i.e. coming from $\mathbb{C}$), then when is the "reduction" of χ modulo some prime p an irreducible character over some characteristic p field?

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If G is solvable, you're getting into Fong-Swan theory and "lifts".

A slightly modified question: If φ is a character of G from a field of characteristic p, then when does there necessarily exist an ordinary irreducible character χ that "reduces" to φ?

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The Fong-Swan theorem for solvable groups says that if G is solvable, then every irreducible character φ of G (from an algebraically closed field of characteristic p) is the "reduction" of an ordinary irreducible character χ of G. We say χ is a **lift** of φ.

My dissertation gave bounds on the number of lifts in a solvable group.

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In S_n, not every characteristic p representation has a lift.

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A very challenging problem: Determine a "generating function" for the number of partitions of n that have no bad boxes.

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$$f(x) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}.$$

What's so special about this function?

If we expand the function (ignoring all issues of convergence - we're working strictly formally here), we see that the coefficient of x^n is exactly the number of partitions of n.

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For instance, the partition $\{5, 3, 2, 2, 2, 2, 1, 1, 1\}$ of 19 corresponds to x^{19} coming from

$$(x^5)(x^3)(x^2)^4(x^1)^3.$$

Is there any hope of determining the generating function for the partitions of n that do not have any bad boxes for some fixed prime p?

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In other words, can we find some function $g(x)$ such that

$$g(x) = \sum_{n=1}^{\infty} p_{irr}(n) x^n,$$

where $p_{irr}(n)$ is the number of partitions of n that have no bad boxes?

This is an ongoing project, and I've made some progress with an undergraduate research assistant by working with a recursive formula and doing some messy calculations on the computer.