1. Let \( n \) be a positive integer, let \( F \) be a field, and suppose \( \alpha \in F \). For \( i, j \) with \( 1 \leq i, j \leq n \), define the matrix \( X_{ij}(\alpha) \) to be the matrix with \( x_{kk} = 1 \) for all \( k, 1 \leq k \leq n \), \( x_{ij} = \alpha \), and all other entries are zero. Prove that \( X_{ij}(\alpha) \) has determinant one.

2. If \( n \) is a positive integer and \( F \) is a field, show that \( \text{SL}(n, F) \) is a normal subgroup of \( \text{GL}(n, F) \). Also, if \( F \) has \( q \) elements, show that
\[
|\text{SL}(n, F)| = q^{n^2-n} \prod_{i=2}^{n} (q^i - 1).
\]

3. Prove that the center of \( \text{SL}(n, F) \) is the set of all diagonal matrices \( cI \) such that \( c^n = 1 \) in \( F \). (This is just like we did in class with \( \text{GL}(n, F) \).)

4. Prove that \( \text{PSL}(2, 3) \) is isomorphic to \( A_4 \) as follows:
   (a) Show that \( \text{SL}(2, 3) \) has order 24 and has 4 Sylow 3-subgroups (You can probably do this by counting elements of order 3).
   (b) Show that if \( \text{SL}(2, 3) \) acts by conjugation on the set of Sylow 3-subgroups, then the kernel of this action is \( \{I, -I\} = Z(\text{SL}(2, 3)) \). Therefore there is an isomorphism of \( \text{SL}(2, 3) \) to a subgroup of \( S_4 \).
   (c) Show that \( A_4 \) is the only subgroup of order 12 in \( S_4 \), and thus \( A_4 \) must be the subgroup in part (b).

5. Let \( X_{i,j}(\alpha) \) be a transvection. Prove:
   (a) \( X_{i,j}(\alpha) \) is in the Borel subgroup \( B \) if and only if \( i < j \).
   (b) \( X_{i,j}(\alpha)X_{j,i}(\beta) = X_{i,j}(\alpha + \beta) \), and thus the set \( \{X_{i,j}(\alpha)|\alpha \in F\} \) forms a subgroup of \( \text{SL}(n, F) \) isomorphic to \( F \).
   (c) Suppose \( k \neq j \). Then \( X_{i,j}(\alpha)e_k = e_k \), where \( e_k \) is the \( k \)th standard column vector (i.e. \( e_k \) has 1 in the \( k \)th row and zero elsewhere).
   (d) \( X_{i,j}(\alpha)e_j = e_j + \alpha e_i \).
   (e) If \( g \in \text{GL}(n, F) \), then the \( i \)th row of \( X_{i,j}(\alpha)g \) is the sum of the \( i \)th row of \( g \) and \( \alpha \) times the \( j \)th row of \( g \). Moreover, the \( k \)th row (if \( k \neq j \)) of \( X_{i,j}(\alpha)g \) is the \( k \)th row of \( g \).

6. Let \( M_{i,j} = X_{j,i}(1)X_{i,j}(-1)X_{j,i}(1) \). Show that \( M_{i,j}e_i = e_j, M_{i,j}e_j = -e_i, \) and \( M_{i,j}e_k = e_k \) for \( k \neq i, j \), where here the \( e_i \) are again the standard column basis vectors.

7. Suppose \( i, j, \) and \( k \) are distinct. Show that \( [X_{i,j}(\alpha), X_{j,k}(\beta)] = X_{i,k}(\alpha \beta) \).
GAP problem

**Problem** Use GAP to make a conjecture about the size of the center of $SL(n,q)$, where $q$ is a power of a prime.