Math 611
HW 1: Due Thursday, January 28th

IMPORTANT: Unless otherwise specified, all groups will be assumed to be finite.

1. Find all of the subgroups of $D_8$ (hint: there are nine of them). Which of them are normal in $D_8$?

2. Show that if $G$ is a finite group and $H$ is a subgroup of $G$ with index two (so $|G : H| = 2$), then $H$ is normal in $G$. Give an example (using $S_3$) to show that this conclusion is not true if 2 is replaced by 3.

3. Suppose $H$ is a subgroup of $G$ and $N \triangleleft G$. Show that $HN$ is a subgroup of $G$.

4. Suppose $H$ is a subgroup of $G$. Show that $H \triangleleft G$ if and only if $xHx^{-1} \subseteq H$ for all $x \in G$.

5. Decompose $\mathbb{Z}_{16}$ and $\mathbb{Z}_{12}$ into simple groups like we did in class.

6. For a group $G$ and a subgroup $H$, define $N_G(H)$ by

$$N_G(H) = \{x \in G|xH = Hx\} = \{x \in G|xHx^{-1} = H\}.$$ 

Show that $N_G(H)$ is a subgroup of $G$ and that $H \triangleleft N_G(H)$.

7. For a group $G$ and a subgroup $H$, define $C_G(H)$ by

$$C_G(H) = \{x \in G|xh = hx \text{ for all } h \in H\}.$$ 

Show that $C_G(H)$ is a subgroup of $N_G(H)$ and that in fact $C_G(H) \triangleleft N_G(H)$.

8. Suppose $\theta : G \rightarrow H$ is a surjective (onto) homomorphism. Show that $\theta$ is an isomorphism if and only if $\ker(\theta) = 1$.

9. Let $H$ be a subgroup of $G$. Show that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$. (Hint: construct a natural function from $N_G(H)$ to $Aut(H)$ and show that the function is a homomorphism and that the kernel is precisely $C_G(H)$.

Don’t forget the GAP problem on the next page!
GAP problem

Begin by downloading GAP. On a PC, this can be done by going to the page http://www.math.colostate.edu/~hulpke/CGT/education.html (there are lots of other places you can find it online, too). There are Mac and Linux and probably Unix versions out there somewhere too. Install GAP (it’s easy - if I can do it, anyone can do it). It comes with a good instruction file.

Problem Use GAP to make a conjecture about the center of the dihedral group $D_{2n}$. Recall that the center of a group $G$ is the set

$$\{x \in G | xg = gx \text{ for all } g \in G\}.$$ 

Do this by having GAP compute the center of various dihedral groups and trying to find a pattern, and then verifying that pattern with more examples.

How? Well, I don’t want to give too much away, because the point of these problems is to play around with GAP yourself. But there is a section in the manual about “Group Libraries”, which should tell you how to “get” the dihedral group, and there are instructions in there about how to find the center of a given group.