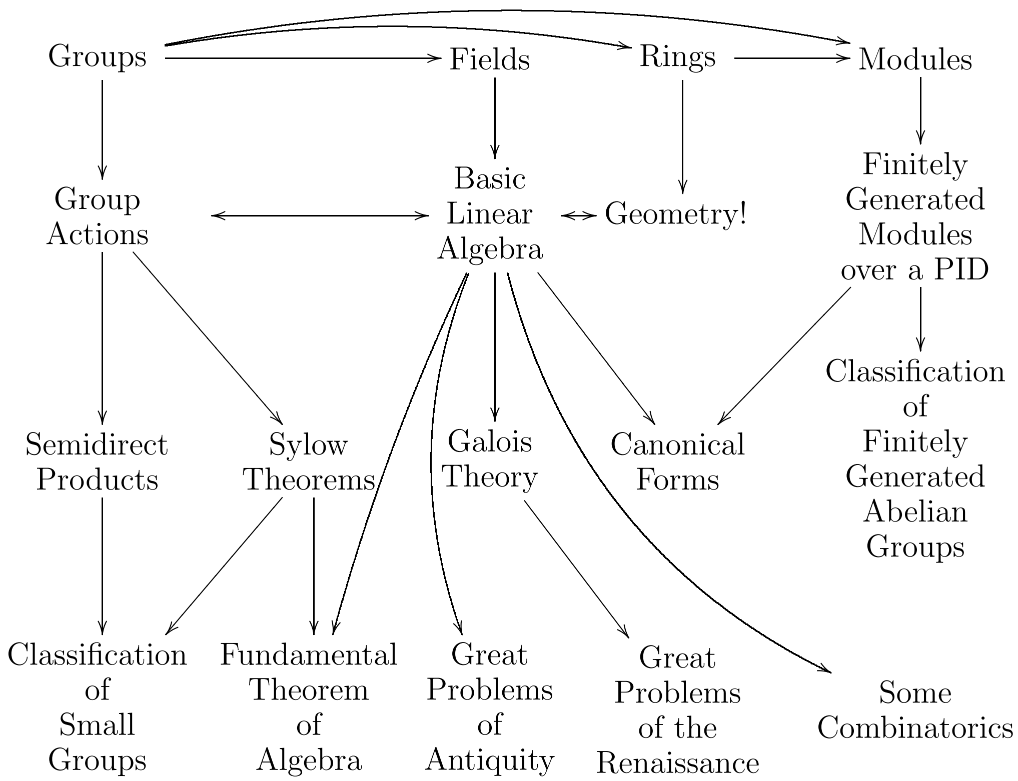
3450:611 Topics in Algebra
Department of Mathematics and Computer Science
The University of Akron

Offered Fall, 2001 (tentative). Instructor: Jeff Adler (`adler@uakron.edu`, 330-972-6779).

Prerequisite: *Abstract Algebra I* and *II*, or permission. I will assume only what was actually covered. **If interested,** please contact me and tell me which parts of algebra you have liked best so far. (If you wish, you may also include any thoughts about the topics described below.) This will partially determine what we cover.

Purpose: Taking up where *Abstract Algebra I* and *II* left off in 2000–01, we will study the most important algebraic objects and some of their applications. The material will be interconnected in several ways, allowing us to apply several areas of algebra to each other. Once we develop a tool, we will apply it as much as possible.

Here is a diagram showing the dependence relationships between several different topics. We will cover a subset of these, to be determined by your interests.



Here are further details on some of the topics mentioned above.

Group actions. When you first learn about groups, you learn that a group is a set, together with a binary operation, such that some conditions are satisfied. However, this is not how groups arise in practice. They arise as ways of acting on things. (For example, the symmetric group S_n acts on a set of size n via permutations. The dihedral group D_n (or D_{2n}; it's a matter of notation) acts on the regular n-gon via rigid transformations. The group of invertible, 2×2 real matrices acts on the plane via linear transformations.) Some theorems, constructions, and applications involving groups become easier once we start thinking of groups in this way. For example...

Sylow Theorems. Even if you have seen these before, we'll do them again. I want to convince you that, though these are major theorems, they are not difficult. You will find it a satisfying experience to learn to reproduce their proofs.

Semidirect Products. Given two groups G and H, you probably know how to form the product group $G \times H$. But if G acts on H, then we can associate to this action a "semidirect" product $G \ltimes H$. This allows you to construct many groups more easily than you could before. For example...

Classification of Small Groups. Once you understand Sylow's Theorems and semidirect products, you can easily construct (up to isomorphism) all groups of order p, p^2, and pq, where p and q are distinct primes. It is just as easy to dispose of orders 18, 20, 28, and others. With only a little more work, you can also handle the cases of p^3 and 12. In particular, you can construct all groups of order up to 31, with the exception of orders 16 and 24.

Basic Linear Algebra. You have already studied linear algebra, but not like this! First of all, we'll drop the usual prejudice that favors the real numbers over other fields. Second, we'll drop any particular concern with matrix manipulation and collections of n-tuples, instead focusing our attention on the objects they really represent: linear transformations and vector spaces.

Modules. These are the basic objects of commutative algebra. They represent a simultaneous generalization of all of the following: modular arithmetic, abelian group theory, ring theory, linear algebra, group representation theory, etc.

Great Problems of Antiquity. Even if you, like me, were not terribly interested in straightedge and compass problems in high school, you might be fascinated by the way in which a little field theory allows you to show easily that several great problems of antiquity are unsolvable. For example:

- Square the circle: Construct a segment whose length is equal to the circumference of a given circle.
- Duplicate the cube: Construct a cube whose volume is double that of a given cube.
- Trisect an arbitrary angle.

Great Problems of the Renaissance. The Babylonians knew how to solve quadratic equations in one variable via formulas that involve only field operations and radicals. Is there such a formula for the solution of a cubic equation? How about a quartic equation? Quintic? These questions motivated the invention of group theory, which solves them.

Fundamental Theorem of Algebra. This theorem states that every polynomial in one variable with complex coefficients has a complex root. If you've taken a course in complex variables, then you've already seen a proof of this. However, Sylow's theorems and basic field theory yield another proof, one which is astonishingly quick.

Combinatorics. We'll study a few striking applications of linear algebra over finite fields.