The Area of a Triangle Using Its Semi-perimeter and the Radius of the In-circle: An Algebraic and Geometric Approach

Lesson Summary:
This lesson is for more advanced geometry students. In this lesson, the students will algebraically prove that the area of a triangle is equal to its semi-perimeter times the radius of its in-circle (A=sr). Then, using Cabri Geometry II, the students will use rotations and translations to transform the triangle into a rectangle. They will then show that the area of the resulting rectangle is equal to the area of the original triangle.

Keywords:
In-circle, semi-perimeter, area of triangle

Existing Knowledge:
Students should have previous knowledge in constructing the in-center and in-circle of a triangle using Cabri Geometry II.

NCTM Standards:
Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.

Learning Objectives:
Students will prove algebraically that the area of a triangle is equal to its semi-perimeter times the radius of its in-circle (A = sr)
Students will use Cabri Geometry II to transform a triangle into a rectangle that has an area equal to the area of the triangle.

Materials:
Cabri Geometry II or Geometer’s Sketchpad

Procedure:
Students should complete Lab 1. The teacher could then have a classroom discussion about the results. Then the students should use Cabri Geometry II to complete Lab 2.
Activity Goals:
In this activity you will algebraically prove that the area of a triangle is equal to its semi-perimeter times the radius of the inscribed circle (A=sr) Then in Laboratory Two, you will use Cabri Geometry II to transform the triangle, using rotations and translations, into a rectangle.

Laboratory One

Notes:
1. This lab involves using in-centers and in-circles of triangles. If you need a review on these topics, refer to the “In-centers and In-circles” lab.
2. The semi-perimeter of a triangle, s, is the same as half the perimeter, p, of the triangle.

Let $D$ be the center of the in-circle of $\triangle ABC$. Let $p$ be the perimeter of $\triangle ABC$. Let $s$ be the semi-perimeter of $\triangle ABC$. Let $r$ be the radius of circle $D$. Prove that the area of $\triangle ABC$ is equal to the length of its semi-perimeter times the radius of circle $D$. (A=sr)

1. What is the relationship between the radii of circle $D$ and the tangent segments $AB$, $BC$ and $AC$?

2. What can you conclude about the six triangles formed?

3. Are $\triangle AGD$ and $\triangle AED$ congruent? Why or why not?
4. Would this also be true for the other pairs of triangles shown? Why or why not?

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5. Why would the area of \( \triangle ABC \) be equal to the sum of the areas of the six triangles?

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6. Write an equation for the area of \( \triangle ABC \) using the sum of the areas of the six right triangles formed (in terms of \( a, b, c, \) and \( r \)).

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7. Using the space provided, simplify and factor your above equation and write the resulting equation here.

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8. What is the relationship of \((a + b + c)\) to the perimeter of \( \triangle ABC \)?

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9. In your own words, what is the area of \( \triangle ABC \)?

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Laboratory Two

1. **Construct $\triangle ABC$**.

2. Construct the angle bisector of $\angle A$, $\angle B$ and $\angle C$, and label their intersection point $D$.

3. Construct $\overline{AD}$, $\overline{BD}$ and $\overline{CD}$ and then hide the angle bisectors.

4. Construct perpendicular lines from point $D$ to sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ and label the points of intersection as $E$, $F$, and $G$, respectively.

5. Construct $\overline{DE}$, $\overline{DF}$ and $\overline{DG}$, then hide the perpendicular lines.

6. Construct a circle with center $D$ and radius $\overline{DE}$.

7. Construct the midpoint of segments $\overline{AD}$, $\overline{CD}$ and $\overline{BD}$.

8. **Construct $\triangle AGD$, $\triangle CGD$ and $\triangle BFD$**.

9. Make a numerical edit of 180 degrees.

10. Rotate $\triangle AGD$ using the 180 degrees about the midpoint of $\overline{AD}$. Name the rotated triangle as $\triangle ADH$.

11. Rotate $\triangle CGD$ using the 180 degrees about the midpoint of $\overline{CD}$. Name the rotated triangle as $\triangle CDI$.

12. Rotate $\triangle BFD$ using the 180 degrees about the midpoint of $\overline{BD}$.
Name the rotated triangle as $\triangle BDJ$.

13. Construct polygon $BFDJ$ and then hide $\triangle BDJ$.


15. Hide polygon $BFDJ$ point $J$, and vector $DI$.

16. Measure $\angle KID$ and rotate polygon $IKLM$ using the measure of $\angle KID$ about point $m$. Name the new polygon clockwise $INOP$.

17. Hide polygon $IKLM$ and points $K$, $L$, and $M$. Also hide the angle measure.

18. Construct vector $NI$ and translate polygon $INOP$ using vector $NI$. Name the translated polygon clockwise $IPQR$.

19. Hide polygon $INOP$, vector $NI$, and points $N$ and $O$.

20. Construct vector $IC$ and translate polygon $IPQR$ using vector $IC$. Name the new polygon $CIRS$.

21. Hide polygon $IPQR$, points $P$ and $Q$, and vector $IC$.

22. Construct polygon $AHRS$ and find its area. Also find the area of $\triangle ABC$. [polygon tool and area tool]

23. What is the relationship of the area of polygon $AHRS$ and the area of $\triangle ABC$?

24. Grab and move point $A$. Do the areas always stay equal? If not, when are they different?
25. The rectangle below is the transformation of the given \( \triangle ABC \) above. Mark all the segments of the rectangle with the appropriate lengths.

26. What is the equation for the area of rectangle \( AHRS \)?

27. The equation for the area of \( \triangle ABC \) was completed in Lab 1. What do you notice about the equation for the area of rectangle \( AHRS \) and the equation for the area of \( \triangle ABC \)?