Transformations and Matrices

Lesson Summary:
Students will explore transformations using matrices and scaling. There are four activities and an appendix. Activity one covers the identity matrix and scaling. Activity two is the linear representation of translations. Activity three is the linear representation of rotations, and activity four is reflections.

Key Words:
Transformations, translations, rotations, reflections

Background Knowledge:
- Able to use Cabri II to create reflections, translations and rotations.
- Knowledge of the four basic construction techniques using compass and straightedge.

Learning Objective:
Discover by the use of Cabri II:
1. The identity matrix
2. The relationship between matrices and rotations, reflections, and translations.
3. Students should be encouraged to take advantage of Cabri’s ability to move objects and observe what happens to measurements, coordinates, etc. when these objects are moved!

Materials:
- Cabri II
- Worksheets

Procedure:
- Group in teams depending upon availability of computers and number in class. (Probably groups of 4 or 5) Make sure at least one person is comfortable with Cabri and/or technology in each group.
- Divide the lesson into 2 labs:
  - Lab 1-Composition of Reflections over two parallel lines is a translation
  - Lab 2-Composition of Reflections over two intersecting lines is a rotation
  - Homework assignment after each lab. To organize the data so that it is ready to submit. Each member of the team submits his or her own data.
- Material needed
  - Copy of each lab for each team member.
  - Computers with Cabri II.
- Assessment:
  - Group Grade for accuracy and thoroughness.
  - Individual grade for work beyond the group.

Linear Transformations and Matrices
Activity One: Identity and Scaling
Team members’ names: ________________________________________________

File name: __________________________________________________________

Goal:
1. Students will be able to use matrix multiplication to perform linear transformations.
2. Students will be able to predict what a certain matrix will do to a given triangle.

Remember: \[
\begin{bmatrix} x & y \\ \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax + cy & bx + dy \end{bmatrix}
\]

Investigate using Cabri:

1. Define grid. [Use grid tool]

2. Create three points with integer coordinates, A, B, and C. [Use point tool]

3. Define Triangle Δ ABC [Use triangle tool]

*On a separate sheet of paper

4. Place the coordinates of Δ ABC in a in a 3x2 matrix. We will call this matrix M.

5. Multiply your matrix M by the matrix I=

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Result: ______________________________________________________________

6. What relationship exists when matrix M is multiplied by matrix I?
The matrix \[
\begin{bmatrix}
1 & 0 \\
0 & 1 
\end{bmatrix}
\] is known as the **identity matrix**.

7. Multiply matrix M by the matrix \[
\begin{bmatrix}
2 & 0 \\
0 & 2 
\end{bmatrix}
\]. Label this matrix M'.

Result: __________________________________________________________


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The original triangle you created is known as the **preimage**. Your new triangle created by the matrix multiplication is known as the **image** of your preimage.

*Back to Cabri*

9. Use the coordinates from matrix M' and create your new image of \( \Delta A'B'C' \) on Cabri. Define the coordinates.  

[Use triangle tool]  
[Use coordinate tool]

10. Compare the lengths of corresponding sides to \( \Delta ABC \) and \( \Delta A'B'C' \). Was your prediction right? Explain  

[Use measurement tool]

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11. Can you generalize your findings for this transformation?

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This is a special type of linear transformation called, **dilation**.

*On paper*

12. Multiply your matrix M by the matrix \[
\begin{bmatrix}
3 & 0 \\
0 & 5 
\end{bmatrix}
\].

Result: __________________________________________________________

13. What happens to the x values? Y values?

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Using Cabri:
14. Define your new $\Delta A''B''C''$ on your Cabri grid. [Use triangle tool]

15. From what you have discovered above, can you formulate a matrix that will half your x-values, and double your y-values?

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16. Can you generalize your findings?

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Note: If students are not familiar with Cabri tools, press F1 on keyboard. A help menu for each tool selected will appear on the bottom of the screen.

Linear Transformations and Matrices

Activity Two - Linear Transformations: Translations

Team members’ names: ______________________________________________________
Goal:
1. Students will discover the properties of translations in a plane.

Investigate using Cabri:

1. Show the axes. [Use axis tool]
2. Create a vector. Place it on the x-axis, label it DF. [Use vector tool]
3. Measure the vector. [Use measurement tool]
4. Draw $\triangle ABC$. Define the coordinates of $\triangle ABC$. [Use triangle tool, coordinate tool]
5. Translate $\triangle ABC$ by the vector DF. Label this triangle A'B'C', and define the coordinates. [Use coordinate tool, translation tool]
6. What do you notice about the relationship between the vertices of $\triangle ABC$ and $\triangle A'B'C'$?

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7. What would happen if we translated our new $\triangle A'B'C'$ by a vector on the y-axis?

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8. Create a vector on the y-axis. Label it vector DG. Measure the vector. [Use vector tool, measurement tool]
9. Translate $\triangle A'B'C'$ by the vector DG. Label this triangle A'B'C''. [Use translation tool]
10. Connect the corresponding vertices of ΔABC and ΔA'B'C' using the segment tool. Measure each segment.  

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12. Repeat steps 9, 10, and 11 with ΔA'B'C' and ΔA''B''C''.  
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13. Draw a line through AB.  

14. Draw a parallel line to AB and through point A'.  

15. What did you discover? Do you believe your findings would hold for all the corresponding segments of your three triangles. Explain.  
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16. List at least three properties of translations that you have discovered in this lab.  
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*Extension:
1. Using the information you have discovered above, can you generalize a formula for the coordinates of vertices of ΔA'B'C' to the corresponding coordinates of the vertices of ΔABC?  
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Note: If students are not familiar with Cabri tools, press F1 on keyboard. A help menu for each tool selected will appear on the bottom of the screen.

Linear Transformations and Matrices  
Activity Three - Linear Transformations: Reflections  

Team members’ names: ___________________________________________________
Goal:
1. Students will be able to use matrix multiplication to perform reflections.
2. Students will discover the matrices that will reflect a triangle across the y-axis and the x-axis.

Remember: 
\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
=
\begin{bmatrix}
  ax + cy \\
  bx + dy
\end{bmatrix}
\]

Identity matrix = 
\[
\begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix}
\]

Investigate using Cabri:

1. Define the axes. Show grid.  
   [Use axis tool]  
   [Use grid tool]

2. Draw a circle centered at the origin and passes through a grid point.  
   [Use circle tool]

3. Create a point on the grid and the circle in the 1st quadrant. Label it point A  
   [Use point tool]

4. Create a triangle using point A as one vertex and two other grid points as the other vertex.  
   [Use triangle tool]

5. Label the other two vertexes B and C. Determine the coordinates of your \( \Delta \) ABC.  
   [Use label tool]

6. Reflect your \( \Delta \)ABC across the y-axis. Label this \( \Delta \) A'B'C'  
   [Use the reflection tool]
As you can see from your picture on Cabri, reflections have the property of **isometry**. That is, the linear transformation preserves measurement. The lengths of the corresponding segments of the two triangles are equal, as are the corresponding angles.

7. Define the coordinates of \( \Delta A'B'C' \). What relationship exists between the coordinates of \( \Delta A'B'C' \) and \( \Delta ABC \)? Explain.

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*On a separate sheet of paper.*

8. Place the coordinates of your original \( \Delta ABC \) in a 3x2 matrix. Call this matrix M.

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9. Can you form a transformation matrix that will reflect your original \( \Delta ABC \) to \( \Delta A'B'C' \)? Call this matrix R. (Hint: Remember what you discovered in step 7, and also the identity matrix.)

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10. Multiply your matrix M by your 2x2 transformation matrix R you came up with in step 9.

Result: ___________________________________________________________

11. Compare the coordinates of your new 3x2 matrix \( M' \) with the coordinates of \( \Delta A'B'C' \). Was your hypothesis transformation matrix R correct? Why or why not?

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12. Now that you know what occurs when you reflect your \( \Delta ABC \) across the y-axis, can you predict what will happen when you reflect it across the x-axis? Explain.

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13. Create a transformation matrix T as you did in step 9 that will reflect your matrix across the x-axis.

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14. Multiply your matrix M by your new transformation matrix T.

Result: ___________________________________________________________
15. Examine your new coordinates. Do you believe you used the correct transformation matrix? Explain why or why not.

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*Back to Cabri.*

16. Reflect your original \( \Delta ABC \) across the x-axis. Label the new triangle \( \Delta A''B''C'' \).

[Use reflection tool]

17. Determine the coordinates of \( \Delta A''B''C'' \).

[Use coordinate tool]

18. Compare the coordinates of \( \Delta A''B''C'' \) with the coordinates you found using your transformation matrix \( T \). Are they the same? If they are, explain why. If they are not, attempt to find the correct transformation matrix that will give you the correct coordinates of a reflection across the x-axis.

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19. Summarize your findings from the lab below. What transformation matrix will give you the coordinates of a reflection across the y-axis, the x-axis?

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*Note:* If students are not familiar with Cabri tools, press F1 on keyboard. A help menu for each tool selected will appear on the bottom of the screen.

**Linear Transformations and Matrices**  
*Activity Four - Linear Transformations: Rotations*

Team members’ names: ________________________________
Goal:
1. Students will be able to use matrix multiplication to perform reflections.
2. Students will discover the matrices used to rotate a triangle around the axis.

Remember: 
\[
\begin{bmatrix}
    x & y
\end{bmatrix}
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} =
\begin{bmatrix}
    ax + cy \\
    bx + dy
\end{bmatrix}
\]

Investigate using Cabri:

1. Define axes and grid. [Use axis tool] [Use grid tool]

2. Draw a circle centered at the origin and passes through a grid point. [Use circle tool]

3. Create a point on the grid and the circle in the 1st quadrant. Label it point A. [Use point tool]

4. Create a triangle using point A as one vertex and two other grid points as the other vertex. [Use triangle tool]

5. Label the other two vertices B and C. Determine the coordinates of your \( \Delta \) ABC. [Use label tool] [Use coordinate tool]

6. Place the number 90 on your screen [Use numerical edit tool]

7. Rotate \( \Delta \) ABC around the origin by 90°. [Use rotation tool]

8. Define the coordinates of \( \Delta A'B'C' \). What relationship exists between the coordinates of \( \Delta A'B'C' \) and \( \Delta ABC \)? Explain.
9. If you were to do another 90° rotation using Δ A'B'C' and the origin, could you predict what the new coordinates of Δ A''B''C'' would be?

Prediction:______________________________________________
______________________________________________________

*On a separate sheet of paper*

10. Place the coordinates of your original Δ ABC in a 3x2 matrix. We will call this matrix M.
________________________________________________________________

11. Can you form a transformation matrix that will rotate your original Δ ABC to Δ A'B'C'? Call this matrix R. (Hint: Remember what you discovered in step 8, and also the identity matrix.)
________________________________________________________________
______________________________________________________

12. Multiply your matrix M by your matrix R you formed in step 11.

Result: _________________________________________________

13. Compare the coordinates of your new 3x2 matrix M' with the coordinates of Δ A'B'C'. Was your hypothesis transformation matrix R correct? Why or why not?
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________________________________________________________________
______________________________________________________

14. Multiply the coordinates of Δ A'B'C' by your 90° rotation matrix R.

Result: _________________________________________________

15. How does your result compare to your predictions from step 9?
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________________________________________________________________
______________________________________________________

*Back to Cabri*

16. Rotate your Δ A'B'C' by 90° around the origin. Label the new triangle Δ A''B''C''.
[Use rotation tool]
17. Determine the coordinates of $\triangle A'B'C'$. [Use coordinate tool]

18. Compare the coordinates of $\triangle A'B'C'$ with the coordinates you found using your transformation matrix $R$. Are they the same? If they are, explain why. If they are not, attempt to find the correct transformation matrix that will give you the correct coordinates of a rotation of $90^\circ$.

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*Extension

In the above lab, we just used $90^\circ$ rotations. You discovered the transformation matrix that will rotate any triangle by $90^\circ$. What if you wanted to rotate your original $\triangle ABC$ by $180^\circ$, or $270^\circ$? What would the transformation matrices be for these rotations. (Hint: Use the information you discovered doing the $90^\circ$ rotations and show how the coordinates relate to the original $\triangle ABC$.)

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\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix} \]

Note: If students are not familiar with Cabri tools, press F1 on keyboard. A help menu for each tool selected will appear on the bottom of the screen.

Matrix Multiplication and Transformation Matrices
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} = \text{Identity matrix}
\]
\[
\begin{bmatrix}
-1 & 0 \\
0 & -1 \\
\end{bmatrix} = 90^\circ \text{ rotation}
\]
\[
\begin{bmatrix}
a & 0 \\
0 & 1 \\
\end{bmatrix} = \text{A scale change in the x direction}
\]
\[
\begin{bmatrix}
-1 & 0 \\
0 & 1 \\
\end{bmatrix} = 180^\circ \text{ rotation}
\]
\[
\begin{bmatrix}
1 & 0 \\
0 & a \\
\end{bmatrix} = \text{A scale change in the y direction}
\]
\[
\begin{bmatrix}
0 & -1 \\
1 & 0 \\
\end{bmatrix} = 270^\circ \text{ Rotation}
\]
\[
\begin{bmatrix}
a & 0 \\
0 & a \\
\end{bmatrix} = \text{A scale change in the x and y direction} \\
y=x
\]
\[
\begin{bmatrix}
a & 0 \\
0 & b \\
\end{bmatrix} = \text{Unequal scaling change}
\]
\[
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta \\
\end{bmatrix} = \text{General rotation about origin}
\]
\[
\begin{bmatrix}
1 & 0 \\
a & 1 \\
\end{bmatrix} = \text{Shear in x direction}
\]
\[
\begin{bmatrix}
1 & b \\
0 & 1 \\
\end{bmatrix} = \text{Shear in y direction}
\]
\[
\begin{bmatrix}
1 & 0 \\
0 & -1 \\
\end{bmatrix} = \text{Reflection across x-axis}
\]
\[
\begin{bmatrix}
-1 & 0 \\
0 & 1 \\
\end{bmatrix} = \text{Reflection across y-axis}
\]