GRAPHIC SEQUENCES WITH A REALIZATION CONTAINING A UNION OF CLIQUES

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Abstract. An integer sequence π is said to be *graphic* if it is the degree sequence of some simple graph G. In this case we say that G is a *realization* of π. Given a graph H, and a graphic sequence π we say that π is *potentially H-graphic* if there is some realization of π that contains H as a subgraph. We define $\sigma(H,n)$ to be the minimum even integer such that every graphic sequence with sum at least $\sigma(H,n)$ is potentially H-graphic. In this paper, we determine $\sigma(H,n)$ for the graph $H = K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_k}$ when n is a sufficiently large integer. This is accomplished by determining $\sigma(K_j + kK_2, n)$ where j and k are arbitrary positive integers, and considering the case where $j = m - 2k$ and $m = \sum m_i$.

1. Introduction

Let G be a simple undirected graph, and let $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. We let $\overline{G}$ denote the complement of G. Denote the complete graph on t vertices by K_t and let $d(v)$ denote the degree of a vertex in a graph G. Given any two graphs G and H, their join, denoted $G+H$, is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{gh \mid g \in V(G),\ h \in V(H)\}$.

A sequence of nonincreasing, nonnegative integers $\pi = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a graph G of order n having degree sequence π. In this case, G is said to *realize* π, and we will write $\pi = \pi(G)$. If a sequence π consists of the terms $d_1, \ldots, d_t$ having multiplicities $m_1, \ldots, m_t$, we may write $\pi = (d_1^{m_1}, \ldots, d_t^{m_t})$.

Let $\sigma(\pi)$ denote the sum of the terms in π. Let π be a graphic sequence and let H be a graph. We say that π is *potentially H-graphic* if there is some realization of π that contains H as a subgraph. Define $\sigma(H,n)$ to be the smallest even integer m so that every n-term graphic sequence π with $\sigma(\pi) \geq m$ is potentially H-graphic. As $\sigma(\pi) = 2|E(G(\pi))|$, the problem of determining $\sigma(H,n)$ is a natural variant of the Turán problem.

The following useful theorem from [5] is an extension of a theorem of Rao [14].

Theorem 1.1. *If π is a graphic sequence with a realization G containing H as a subgraph, then there is a realization G' of π containing H with the vertices of H having the $|V(H)|$ largest degrees of π.*

For a survey of results we refer the reader to [15], and for any undefined terms to [1]

2. Known Results

2.1. **Cliques.** In [2] Erdős, Jacobson and Lehel conjectured that

$$\sigma(K_t, n) = (t-2)(2n-t+1)+2.$$

The conjecture arose from consideration of the graph $K_{(t-2)} + \overline{K}_{(n-t+2)}$. It is easy to see that this graph contains no K_t, is the unique realization of the sequence

$$((n-1)^{t-2}, (t-2)^{n-t+2}),$$

and has degree sum $(t-2)(2n-t+1)$. Erdős *et al.* proved the conjecture for $t = 3$ and $n \geq 6$. The cases $t = 4$ and 5 were proved separately (see [5] and [8], and [9]). Li, Song and Luo [10] proved the conjecture true for $t \geq 6$ and $n \geq \binom{t}{2} + 3$. The following summarizes these results.

Theorem 2.1. *For $t \geq 3$ and $n \geq n(t)$ sufficiently large,*

$$\sigma(K_t, n) = (t-2)(2n-t+1)+2.$$

The goal of this paper is to extend Theorem 2.1 as follows:

Theorem 2.2. *Let $H = K_{m_1} \cup \cdots \cup K_{m_k}$ where each $m_i \geq 2$. Then for sufficiently large n,*

$$\sigma(H, n) = (m-k-1)(2n-m+k)+2,$$

where $m = \sum_{i=1}^{k} m_i$.

One can see that $\sigma(H, n) > (m-k-1)(2n-m+k)$ by considering the graphic sequence

$$\pi = ((n-1)^{m-k-1}, (m-k-1)^{n-m+k+1}) \tag{1}$$

which is uniquely realized by $K_{m-k-1} + \overline{K}_{n-m+k+1}$. This graph does not contain a copy of H, as this would force two vertices from the same clique to reside in $\overline{K}_{n-m+k+1}$. The sequence in the case where $m_1 = m_2 = \cdots = m_k$ first appears in [16], and the value of $\sigma(kK_2, n)$ was determined in [5]. Additionally, (1) can be generated using the techniques found in [3].

2.2. **The Friendship Graph.** The friendship graph, denoted F_k, is the graph composed of k triangles intersecting in a single vertex. F_k can also be defined as $K_1 + kK_2$. The following was shown in [4].

Theorem 2.3. *For $k \geq 1$ and $n \geq \frac{9}{2}k^2 + \frac{7}{2}k - \frac{1}{2}$,*

$$\sigma(F_k, n) = k(2n-k-1)+2. \tag{2}$$

In this paper, we will consider the graph F_k^j defined as $K_j + kK_2$, which is an extension of the friendship graph. We will show the following:

Theorem 2.4. *Let j and k be positive integers. Then there exists a positive integer $n(j,k)$ such that for all $n > n(j,k)$,*

$$\sigma(F_k^j, n) = (j+k-1)(2n-j-k)+2.$$

Consideration of the graphic sequence

$$\pi = ((n-1)^{j+k-1}, (j+k-1)^{n-j-k+1}) \tag{3}$$

which is uniquely realized by $K_{j+k-1} + \overline{K}_{n-j-k+1}$, yields that for all $n \geq j+2k$,

$$\sigma(F_k^j, n) > (j+k-1)(2n-j-k). \tag{4}$$

Both (3) and (4) first appeared in [16] and can also be generated using the techniques in [3].

Again letting $H = K_{m_1} \cup \cdots \cup K_{m_k}$ with each $m_i \geq 2$ and $m = \sum_{i=1}^k m_i$, we note that F_k^{m-2k} contains H as a subgraph. This immediately implies that $\sigma(H,n) \leq \sigma(F_k^{m-2k}, n)$. Theorem 2.4 would therefore imply that $\sigma(H,n) \leq (m-k-1)(2n-m+k)+2$ and thus prove Theorem 2.2.

3. Proof of Theorem 2.4

From here forward, let n be a sufficiently large integer and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing, nonnegative integer sequence with $\sigma(\pi) \geq (j+k-1)(2n-j-k)+2$. We will present several facts about the terms of π.

Lemma 3.1. *The following hold:*

(i) $d_j \geq j+2k-1$.
(ii) $d_{j+k+1} \geq j+k$

Proof. To establish (i), assume that $d_j \leq j+2k-2$, and note that this implies

$$\sigma(\pi) \leq (n-1)(j-1) + (n-j+1)(j+2k-2).$$

For n sufficiently large, this is strictly less than $(j+k-1)(2n-j-k)+2$. We establish (ii) by noting that Theorem 2.1 implies that $\sigma(\pi) \geq \sigma(K_{j+k+1}, n)$. This implies that d_{j+k+1} must be at least $j+k$.

□

Furthermore, we may assume that

$$d_n \geq j+k-1. \tag{5}$$

Indeed, let G be some realization of π, and assume that there is some vertex v_0 in $V(G)$ such that $\delta(G) = d(v_0) \leq j+k-2$. We let $G_1 = G - v_0$ and note that

$$\sigma(\pi(G_1)) \geq \sigma(\pi(G)) - 2(j+k-2) \geq (j+k-1)(2(n-1)-j-k)+4.$$

If the minimum degree of G_1 is at least $j+k-1$, we will redefine π to be $\pi(G_1)$. Otherwise we will continue this procedure and generate graphs $G_2, \ldots, G_i$ as follows. If G_{i-1} has minimum degree at least $j+k-1$, then we will redefine π to be $\pi(G_{i-1})$. Otherwise, let v_{i-1} be a vertex in $V(G_{i-1})$ such that $\delta(G_{i-1}) = d(v_{i-1}) \leq j+k-2$. We let $G_i = G_{i-1} - v_{i-1}$ and note that

$$\sigma(\pi(G_i)) \geq \sigma(\pi(G_{i-1})) - 2(j+k-2) \geq (j+k-1)(2(n-i)-j-k)+2+2i.$$

Either this process will terminate, or if n is sufficiently large there will be some sufficiently large i such that $\sigma(\pi(G_i)) \geq \sigma(K_{j+2k}, n-i)$. If such an i exists, then $\pi(G_i)$, and hence $\pi(G)$ will be potentially K_{j+2k}-graphic. This would suffice to prove Theorem 2.4, and hence we may assume that for some G_i, $\delta(G_i) \geq j+k-1$, and redefine π to be $\pi(G_i)$.

We construct the sequence

$$\pi_1 = (d_2^{(1)}, \dots, d_n^{(1)})$$

from π by deleting d_1, reducing the first d_1 remaining terms of π by 1 and then reordering the last $n-j-2k$ terms to be nonincreasing. For $2 \leq i \leq j$, we construct

$$\pi_i = (d_{i+1}^{(i)}, \dots, d_n^{(i)})$$

from

$$\pi_{i-1} = (d_i^{(i-1)}, \dots, d_n^{(i-1)})$$

by deleting $d_i^{(i-1)}$, reducing the first $d_i^{(i-1)}$ nonzero terms of π_{i-1} by 1, and then ordering the last $n-j-2k$ terms to be nonincreasing. The manner in which we construct π_i, $j+1 \leq i \leq j+2k$ depends on two cases.

Case 1: $d_{j+k-1} \geq j+2k-1$.

In this case, we proceed as above and construct π_i, $j+1 \leq i \leq j+2k$ from π_{i-1} by removing $d_i^{(i-1)}$, reducing the first $d_i^{(i-1)}$ nonzero terms by 1, and then ordering the last $n-j-2k$ terms to be nonincreasing.

Case 2: There is some l, $0 \leq l \leq k-2$ such that $d_{j+k-l-1} < j+2k-1$.

Take the maximal such l; thus if $l < k-2$, we assume that $d_{j+k-l-2} \geq j+2k-1$. We construct π_i, $j+1 \leq i \leq j+k-l-2$ as above, by removing $d_i^{(i-1)}$ from π_{i-1}, reducing the first $d_{i+1}^{(i)}$ terms by 1, and then ordering the last $n-j-2k$ terms to be nonincreasing. We then construct π_i, $j+k-l-1 \leq i \leq j+k$ from π_{i-1} by removing $d_i^{(i-1)}$, reducing $d_i^{(i-1)}$ consecutive positive terms *starting with* $d_{j+2k+1-i}$ by one, and then ordering the last $n-j-2k$ terms to be nonincreasing. Note that the assumption that $d_n \geq j+k-1$ implies that $d_n^{(j+k-l-2)} \geq 1$. Thus, since $d_i^{(i-1)} \leq j+2k-1-i$, we can be assured that if n is large enough, say at least $(j+2k)(k+2)$, there will be a sufficient number of positive terms in each π_{i-1} to construct π_i without forcing any term in π_i to be negative. We then construct π_i, $j+k+1 \leq i \leq j+2k$ from π_{i-1} as as above, by removing $d_i^{(i-1)}$, reducing the first $\pi_i^{(i-1)}$ positive terms by one, and then ordering the last $n-j-2k$ terms to be nonincreasing.

We now present a crucial lemma.

Lemma 3.2. *If π_{j+2k} is graphic, then π is potentially $K_j + kK_2$-graphic.*

Proof. We first note that if π_{j+2k} is graphic, we can construct realizations of π_i, $1 \leq i \leq j+2k-1$ and π by adding a vertex to $G(\pi_{i+1})$ and adjoining it to vertices of those degrees that were reduced by one in the formation of π_{i+1}. In creating $\pi_1, \dots, \pi_j$, the fact that $d_j \geq j+2k-1$ implies that the realization of π created in this manner will contain a copy of $K_j + \overline{K}_{2k}$ on the vertices of degree $d_1 \dots, d_{j+2k}$, with the clique lying on those vertices of degree $d_1, \ldots, d_j$. It remains to show that in this realization of π it is also possible to construct a matching of size k on the vertices of degree $d_{j+1}, \ldots, d_{j+2k}$. To see this, we simply note that in constructing π_{j+i}, $1 \leq i \leq k$ we always reduce $d^{(i)}_{j+2k+1-i}$ by one, assuring that in our realization of π, the vertex of degree d_{j+i} will be adjacent to the vertex of degree $d_{j+2k+1-i}$, assuring the desired matching. □

We will use the following two theorems to complete the proof of Theorem 2.4, found in [11] and [13], respectively.

Theorem 3.3. *Let* $\pi = (d_1, d_2, \ldots d_n)$ *be a non-increasing sequence of non-negative integers, where* $d_1 = m$ *and the degree sum is even. If there exists an integer* $n_1 \leq n$ *such that* $d_{n_1} \geq h \geq 1$ *and* $n_1 \geq \frac{1}{h}\left\lfloor \frac{(m+h+1)^2}{4} \right\rfloor$, *then* π *is graphic.*

Theorem 3.4. *Let* $n \geq 2r+2$ *and* $\pi = (d_1, d_2, \ldots d_n)$ *be graphic with* $d_{r+1} \geq r$. *If* $d_{2r+2} \geq r-1$, *then* π *is potentially* K_{r+1}*-graphic.*

The following two lemmas will complete the proof of Theorem 2.4.

Lemma 3.5. *Let* π *be an* n*-term graphic degree sequence for sufficiently large* n *with degree sum at least* $(j+k-1)(2n-j-k)+2$. *If* $d_{2j+4k} \geq j+2k-2$ *then* π *is potentially* $K_j + kK_2$*-graphic.*

Proof. If $d_{j+2k} \geq j+2k-1$ then π is potentially K_{j+2k}-graphic by Theorem 3.4, which suffices to prove Theorem 2.4. Otherwise, $d_{j+2k} \leq j+2k-2$ and this implies that $d_{j+2k} = d_{j+2k+1} = \cdots = d_{2j+4k} = j+2k-2$, and for $1 \leq i \leq j+2k$, the terms $d^{(i)}_{j+2k+1}, \ldots, d^{(i)}_{2j+4k}$ differ by at most one. Hence π_{j+2k} satisfies, for some $m \geq 1$,

$$j+2k-2 \geq m = d^{(j+2k)}_{j+2k+1} \geq \ldots \geq d^{(j+2k)}_{2j+4k} \geq m-1.$$

If $m = 1$, π_{j+2k} must be graphic as the degree sum of π_{j+2k} is even. If $m \geq 2$, then

$$\frac{1}{m-1}\left\lfloor \frac{(m+(m-1)+1)^2}{4} \right\rfloor \leq m+2 \leq j+2k.$$

By Theorem 3.3 π_{j+2k} is graphic, and hence by Lemma 3.2, π is $K_j + kK_2$-graphic. □

Lemma 3.6. *Let* π *be an* n*-term graphic degree sequence for sufficiently large* n *with degree sum at least* $(j+k-1)(2n-j-k)+2$. *If* $d_{2j+4k} \leq j+2k-3$ *then* π *is potentially* $K_j + kK_2$*-graphic.*

Proof. First, we establish that $d_1 \geq 2j+4k-2$. If not, then

$$\sigma(\pi) \leq (2j+4k-3)(2j+4k-1) + (j+2k-3)(n-2j-4k+1).$$

This is less than $(j+k-1)(2n-j-k)+2$ for n sufficiently large.

We proceed by induction on j. Theorem 2.3 establishes Theorem 2.4 in the case $j = 1$. We thus assume $j \geq 2$.

Now suppose that either $d_1 = n-1$ or there exists an r such that $2k+1 \leq r \leq d_1+1$ such that $d_{r+1} < d_r$. As the degree sum of π_1, which is graphic as a consequence of the Havel-Hakimi algorithm [6, 7], is at least $\sigma(K_{j-1}+kK_2, n-1)$, by induction there is some realization of π_1 that contains a copy of $K_{j-1}+kK_2$. Furthermore, by Theorem 1.1 this implies that there exists a realization of π_1 with $K_{j-1}+kK_2$ on those vertices having degree $d_2^{(1)}, \dots, d_{j+2k}^{(1)}$. This implies that π would be potentially K_j+kK_2-graphic.

Thus we assume that no such r exists and hence that

$$n-2 \geq d_1 \geq d_2 \geq \ldots \geq d_{j+2k} = d_{j+2k+1} = \ldots = d_{2j+4k} = \ldots = d_{d_1+2}.$$

We can thus conclude that there is some $m \geq 1$ with

$$j+2k-3 \geq m = d_{j+2k+1}^{(j+2k)} \geq \ldots \geq d_{2j+4k}^{(j+2k)} \geq m-1.$$

We may then complete the proof as in the previous lemma. □

Together, Lemma 3.5 and Lemma 3.6 complete the proof of Theorem 2.4. As discussed above, this suffices to prove Theorem 2.2.

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REFERENCES

[1] Bollobás, B., *Extremal Graph Theory*, Academic Press Inc. (1978).

[2] Erdős, P., Jacobson, M.S., Lehel, J., Graphs Realizing the Same Degree Sequence and their Respective Clique Numbers, *Graph Theory, Combinatorics and Applications*, Vol. I, 1991, ed. Alavi, Chartrand, Oellerman and Schwenk, 439-449.

[3] Ferrara, M., Schmitt, J., A Sharp Lower Bound for Potentially H-graphic Sequences. submitted.

[4] Ferrara, M., Gould, R., Schmitt, J., Graphic Sequences with a Realization Containing a Friendship Graph, to appear in *Ars Combinatoria*.

[5] Gould, R.J., Jacobson, M.S., Lehel, J., Potentially G-graphic degree sequences, *Combinatorics, Graph Theory, and Algorithms* (eds. Alavi, Lick and Schwenk), Vol. I, New York: Wiley & Sons, Inc., 1999, 387-400.

[6] Hakimi, S.L., On the realizability of a set of integers as degrees of vertices of a graph, *J. SIAM Appl. Math*, **10** (1962), 496-506.

[7] Havel, V., A remark on the existence of finite graphs (Czech.), *Časopis Pěst. Mat.* **80** (1955), 477-480.

[8] Li, J., Song, Z., An extremal problem on the potentially P_k-graphic sequences, *The International Symposium on Combinatorics and Applications*, June 28-30, 1996 (W.Y.C. Chen et. al., eds.) Tanjin, Nankai University 1996, 269-276.

[9] Li, J., Song, Z., The smallest degree sum that yields potentially P_k-graphical sequences, *J. Graph Theory* **29** (1998), no.2, 63-72.

[10] Li, J., Song, Z., Luo, R., The Erdős-Jacobson-Lehel conjecture on potentially P_k-graphic sequences is true, *Science in China, Ser. A*, **41** 1998, (5):510-520.

[11] Li, J., Yin, J., The smallest degree sum that yields potentially $K_{r,r}$-graphic sequences, *Science in China, Ser. A*, **45** (June 2002),(6):694-705.

[12] Li, J., Yin, J., An extremal problem on potentially $K_{r,s}$-graphic sequences, *Discrete Math.*, **260** (2003), 295-305.

[13] Li, J., Yin, J., Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size, to appear in *Discrete Math.*

[14] Rao, A.R., The clique number of a graph with a given degree sequence, Proc. Symposium on Graph Theory (A.R. Rao ed.), MacMillan and Co. India Ltd., *Indian Statistical Institute Lecture Notes Series* **4** (1979), 251-267.

[15] Rao, S.B., A survey of the theory of potentially P-graphic and forcibly P-graphic degree sequences, *Lecture Notes in Math.*, No. **855**, Springer Verlag, 1981, 417-440.

[16] Schmitt, J. On Potentially P-graphic Degree Sequences and Saturated Graphs. Ph.D. Dissertation, Emory University. May 2005.